REPORT RESUMES

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NUMBER AND OPERATION, HANDBOOK FOR ELEMENTARY MATHEMATICS WORKSHOPS.

BY- PHILLIPS, GUSSIE

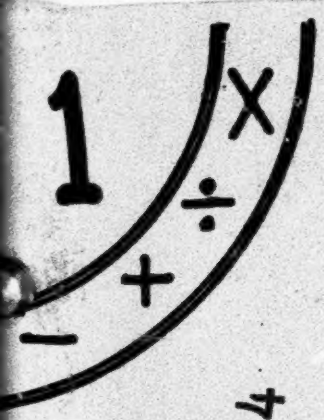
ILLINOIS STATE OFF. SUPT. PUB. INSTR., SPRINGFIELD

PUB DATE 62

EDRS PRICE MF-$1.00 HC-$8.52 211P.

DESCRIPTORS- *ARITHMETIC, *ELEMENTARY SCHOOL MATHEMATICS, *INSERVICE TEACHER EDUCATION, *MATHEMATICS, *TEACHER EDUCATION, MATHEMATICS EDUCATION, WORKSHOPS,

THIS NUMBER AND OPERATION HANDBOOK IS INTENDED TO HELP EMPHASIZE THE SEQUENTIAL NATURE OF MATHEMATICS AND FOCUS ATTENTION ON THE NEED FOR A WELL-PLANNED PROGRAM TO PROVIDE FOR CONTINUITY IN THE DEVELOPMENT OF CONCEPTS IN MATHEMATICS. EMPHASIS IS ON NEW APPROACHES IN TEACHING OF STANDARD TOPICS TO GAIN UNDERSTANDING. DIFFERENT SYSTEMS OF NUMERATION ARE STUDIED, AND THE DEVELOPMENT AND USE OF TERMINOLOGY AND SYMBOLISM IS STRESSED. REPORTED ON ARE--(1) NUMBER AND NUMERALS - AN INTRODUCTION, (2) SYSTEMS OF NUMERATION, (3) SETS AND SENTENCES, (4) OPERATIONS OF ADDITION AND MULTIPLICATION, (5) OPERATION OF SUBTRACTION, (6) OPERATION OF DIVISION, (7) OUR NUMBER SYSTEM - AN OVERVIEW, AND (8) OTHER INTERESTING FACTS ABOUT NUMBERS. EACH OF THE ABOVE UNITS IS ORGANIZED AS FOLLOWS--(1) SUBJECT MATTER AND EXPLANATORY MATERIAL, (2) REFERENCES FOR TEACHERS, AND (3) ACTIVITIES. THE ACTIVITIES AND TEACHING AIDS DESCRIBED IN THE LATTER SECTION ARE INTENDED TO BE USED IN THE CLASSROOM (1) TO DEVELOP CONCEPTS, (2) TO PROVIDE ENRICHMENT AND PROJECT MATERIALS, (3) TO PROVIDE DRILL, AND (4) TO AID IN INTRODUCING OLD TOPICS IN NEW WAYS. (RP)



Math

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NUMBER AND OPERATION

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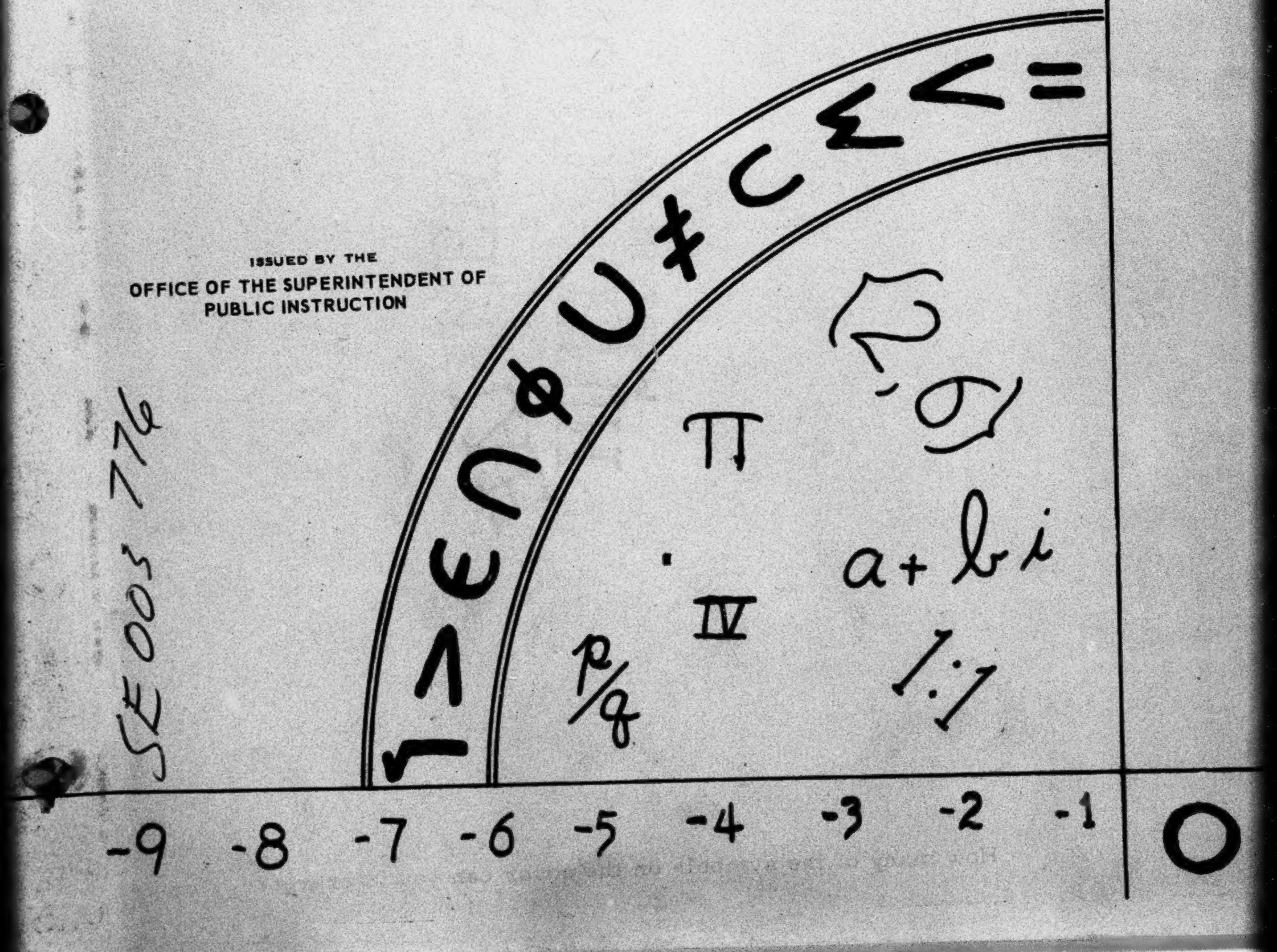

How many of the symbols on the cover can you interpret?

HANDBOOK
for
ELEMENTARY
MATHEMATICS
WORKSHOPS

STATE OF ILLINOIS

The Office of the Superintendent of
Public Instruction

RAY PAGE, Superintendent

This 1962 revision was made by:

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The original materials were prepared in 1960 as an extension of Bulletin C-2, "Thinking in the Language of Mathematics", Illinois Curriculum Program, for use in In-Service Education Workshops conducted in the State of Illinois. The original content and format were planned by a committee consisting of:

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INTRODUCTION

Most people are aware of the close scrutiny that the present mathematics curriculum at all levels is undergoing. A great number of research projects underway throughout the nation have demonstrated that children can learn mathematics at an earlier age and are resulting in a more critical analysis of existing curricula and teaching techniques. Although most of the earlier experimentation was at the secondary level, it has been extended during the past few years to the elementary school where students spend eight important years. This extension helps to emphasize the sequential nature of mathematics and focuses attention more than ever on the need for a well-planned program to provide for continuity in the development of concepts. Some of these basic concepts have their beginnings in the kindergarten.

But why change the curriculum in arithmetic? Isn't the present stimulus-response, computational-centered arithmetic adequate?

Technological advances have been so rapid that no one can predict what problems people will have to solve or what mechanical means for solving problems will exist in the future. This is another basic reason for the clamor for reform. The former concentration on skills appears inadequate. Arithmetic can no longer be considered solely a tool subject but it must be viewed in its larger framework of mathematics structured upon major ideas. A well-developed and well-understood concept can be easily adapted for use even in a world of change.

The reform in mathematics doesn't necessitate a complete upheaval of the subject matter now being taught. The basic content of arithmetic is not being changed a great deal. A few new topics are being introduced, but major emphasis is now on structure, continuity of concepts, and understanding. Discovery procedures are used to develop basic concepts and basic rules; drill then becomes meaningful.

In this workshop, teachers will have an opportunity to identify some of the important aspects of new programs and thus will have some guidelines for choosing a new program or for updating and enriching one presently used. Emphasis will be on new approaches in teaching of standard topics to gain understanding. A new topic--sets--will be introduced. Different systems of numeration will be studied. The development and use of terminology and symbolism will be stressed because mathematics is a language with its own vocabulary. Some of the conflicting views existing in elementary mathematics will be discussed.

Each unit is organized as follows: (1) subject matter and explanatory material, (2) references for teachers, and (3) activities.

Discussions will be based upon the lecture and the printed materials. The SUBJECT MATTER AND EXPLANATORY MATERIAL are intended to supplement the lectures. The margins in this section are left for your notes.

A selected reading list is given at the end of each unit in REFERENCES FOR TEACHERS. Space is provided for you to add notes about any of the books which you find helpful or for you to list additional references.

The activities and teaching aids described in the ACTIVITIES section may be used in the classroom for various purposes: to develop concepts; to provide enrichment and project materials; to afford drill in disguised forms; or to aid in introducing old topics in new ways. The leader may have time to discuss or to demonstrate some at each meeting, but this is primarily your section. Other teaching aids besides the ones mentioned can be constructed easily and space is left for notes.

CONTENTS

UNIT I NUMBERS AND NUMERALS: AN INTRODUCTION

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UNIT I

NUMBERS AND NUMERALS

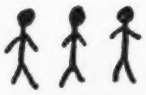
AN INTRODUCTION

UNIT I. NUMBERS AND NUMERALS: AN INTRODUCTION

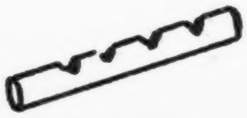
"One, two, plenty"--Tasmanian Method of Counting

EARLY USE OF NUMBERS

From earliest time man has used number. His ideas probably were communicated first by pictures and words. Early methods of keeping count were by scratching notches in sticks, tying knots in strings, and putting pebbles in a heap to represent the objects being counted.



THREE MEN

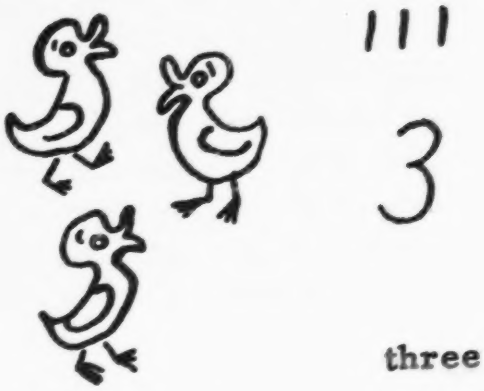


TALLY STICK

ONE-TO-ONE CORRESPONDENCE

The methods of keeping count were simple matching processes where the object and the stroke, knot, or pebble were in one-to-one correspondence.

This one-for-one matching of objects was a forerunner of counting. As soon as man began to match in a definite order he was counting. This meant he had to develop words for his counting system to communicate his ideas, and the next step was the development of symbols to represent these words.



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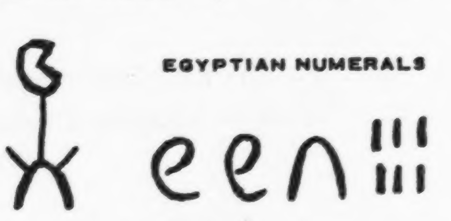
INVENTION OF NUMERALS

Early representations of numbers were quite simple and in many cases mere repetition of a few basic symbols. Observe the representation of 1216 by the ancient Egyptians.



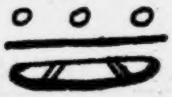
EGYPTIAN NUMERALS

1216

Our present system of numeration (way of representing numbers symbolically) is the Hindu-Arabic decimal system. It uses ten basic symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, called digits. The value of any number represented by a numeral is equal to the sum of the values that the digits represent because of their positions in the numeral. For example, 243 equals 2 hundreds plus 4 tens plus 3 ones. The system is an additive system using positional notation, and '0' is essential when representing numbers like 305.

This numeration system received the name Hindu-Arabic because the numerals were invented in India and brought to Europe by the Arabs. Oddly enough, the Arabs today use different symbols from the ones they gave to the rest of the world.

MAYAN NUMERALS



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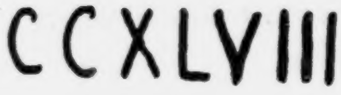
CAN YOU SEE WHY?

It is interesting, too, to note that the word "digit" is an abbreviation of the Latin word "digitus" meaning "finger" or "toe", and the use of this word for the numerals 0-9 is probably based on the common practice of counting with fingers.

ROMAN NUMERALS



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PRECISENESS OF LANGUAGE OF MATHEMATICS

One characteristic of mathematics is the preciseness of its language, and most of the modern programs in mathematics carefully develop correct usage of mathematical vocabulary.

Every branch of learning has a vocabulary of its own, and mathematics is no exception. Definitions of terms in mathematics should always be kept in mind as definitions for this field only, regardless of dictionary definitions in general. For instance, many dictionaries make little or no distinction between words such as number and numeral, and some define them as synonyms.

NUMBERS AND NUMERALS

To a mathematician number represents an idea. It is a way of thinking about individual objects in a set or collection as forming that particular set. A person cannot see, draw, or write a number although he represents his idea of a number by using a symbol called a numeral.

Five books, five ducks, or five cents in a nickel have fiveness in common. The Roman numeral 'V' and the Hindu-Arabic numeral '5' both represent this idea of fiveness.

DISTINGUISHING BETWEEN NUMBER AND NUMERAL

In print, when the reference is to the number idea, the symbol 5 or word five is written without quotes. When the reference is to the numeral or written representation itself, the symbol '5' or word 'five' is written with single quotes. Thus 30, four, and V all represent the number idea while '30', 'four', and 'V' refer to the marks on the paper.

To test your understanding of numbers and numerals, try answering the five questions below before reading the answers and explanations which follow.

1. Is 3 larger than 4 ?
2. Is '6' smaller than '5' ?
3. Does '36' consist of 2 numbers?
4. Does 'four' have four letters?
5. Does four have four letters?

ANSWERS:

1. No. The number 3 is smaller than the number 4.
2. Yes. The symbol or numeral '6' is smaller in size than the symbol '5'.
3. No. A symbol or numeral can't consist of 2 numbers.
4. Yes. The word 'four' consists of the four letters 'f', 'o', 'u' and 'r'.
5. No. A number does not consist of letters.

For additional exercises, consult the 24th Yearbook of the National Council of Teachers of Mathematics--The Growth of Mathematical Ideas--pp. 336-337.

DIFFERENT NAMES FOR THE SAME NUMBER

It is important that number sense be developed early. A child should be taught to think of various ways of representing a number. For example, other names for 9 include '4+5', '6+3', '3^2', '17-8', '18÷2', etc. As a child realizes that a number may have many different names, he develops his sense of number.

Formerly, these ways of representing 9 were thought of only as indicating operations that had to be performed. Today they are recognized also as other names for 9.

NUMBER SYSTEMS AND SYSTEMS OF NUMERATION

Similar to the distinction between number and numeral is the distinction between a number system and a system of numeration.

A system of numeration deals only with the symbols and rules for writing the numerals. In Unit II systems of numeration using bases other than 10 will be examined.

A mathematical system consists of:

1. undefined terms
2. defined terms
3. assumptions or axioms
4. conclusions or theorems

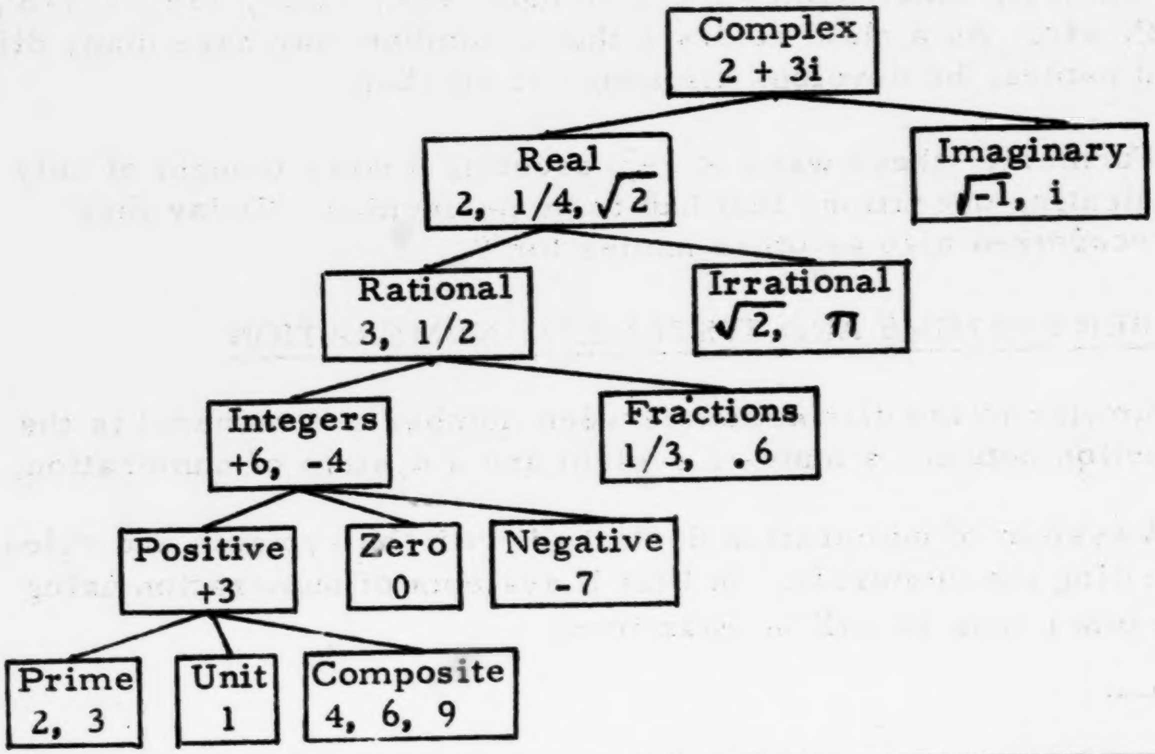
An assumption or axiom is a statement accepted without proof. A theorem or conclusion is a statement that can be formally proved by logical reasoning.

In a mathematical system undefined terms and unproved statements or axioms are agreed upon. Then other terms are defined in terms of the undefined terms, axioms, previously defined terms, and non-mathematical words. Later, other conclusions or theorems are established through logical inferences.

Different assumptions or axioms lead to different mathematical systems. For example, in Euclidean geometry it is assumed that through a given point one line and only one line can be drawn parallel to a given line. Different geometries, such as those now being used for space exploration, result when it is assumed that through a given point no line or perhaps several lines can be drawn parallel to a given line.

The Real Number System is a mathematical system. In unit 4-7 it will be developed using zero and one as undefined terms, and addition and multiplication as undefined operations. The other natural numbers will be defined in terms of one making use of the undefined operations. The operations of subtraction and division will be defined in terms of addition and multiplication. Some authorities consider only addition as an undefined operation and multiplication an operation defined in terms of addition.

These Units will give teachers an organized overview of the numbers used in elementary mathematics. The following diagram shows the relationships to be discussed.



A SET AND ITS ELEMENTS

The language of sets is used in many of the modern programs in mathematics. Although set is an undefined term, it may be considered as any collection of well-defined objects. These objects are called the elements of the set. Some ideas of arithmetic may be clarified through the use of sets.

A child may speak of a set of books or a set of dishes. A mathematician may speak of the set of natural numbers or the set of integers or the set of rational numbers.

Mathematicians consider both finite and infinite sets. A finite set is one in which the number of elements can be counted with the counting coming to an end. An infinite set is one in which the elements cannot be counted with the counting coming to an end.

SET NOTATION

Braces are the notational symbols used to indicate a set. Thus $\{1, 2, 3\}$ would be read as "the set whose elements are the numbers 1, 2, and 3". {Mary, Judy} would be read as "the set whose elements are Mary and Judy". An understanding of this notation is helpful in understanding current books and magazine articles.

NATURAL NUMBERS

Some mathematicians refer to the numbers 1, 2, 3, 4, . . . as the natural numbers. Other mathematicians include zero as a natural number. In these units zero will be considered a natural number in the set N of all natural numbers.

It is assumed that a natural number always has a successor. This means that it is always possible to count one more. That the sequence of natural numbers continues without end (infinite sequence) is indicated by three dots as shown below:

$$N = \{0, 1, 2, 3, 4, 5, \cdots\}$$

CARDINAL AND ORDINAL NUMBERS

A natural number may be used to indicate how many objects are in a set or which member of an ordered set is being considered. When used to indicate "how many" it is called a cardinal number. When used to indicate "which one" it is called an ordinal number.

Man developed early the idea of cardinal number and spoke of sets of two, five, three, etc. with no order implied. He also found need to order numbers and used the idea of ordinal number to refer to object number two, three, or five, etc. in this ordered set. This order is frequently described by the adjectives second, third, fifth, etc.

A group of three children implies the cardinal use of the number 3 since it tells how many children are in the group. Child number 3 in a row implies the ordinal use of the number 3 since it designates a particular child in the row.

Elementary programs have traditionally placed emphasis upon counting. Today, because of the increased use of sets, cardinal numbers are receiving additional emphasis. Pictured exercises aid children in learning to identify the number of the set without counting. A child learns to associate zero with no objects, one with a single object, two with a pair of objects, etc.

CARDINAL NUMBER OF A SET

When the elements of two sets can be put into one-to-one correspondence, the sets are said to contain the same number of elements and to have the same cardinal number. To understand better the idea of the cardinal number of a set, examine the sets below:

The cardinal number 0 is associated with	{ }	a set containing no elements
The cardinal number 1 is associated with	{ * }	a set containing a single element
The cardinal number 2 is associated with	{ *, * } { •, ♦ }	a set containing two elements, alike or different
The cardinal number 3 is associated with	{*, *, *} {▲, •, ■}	a set containing three elements, alike or different

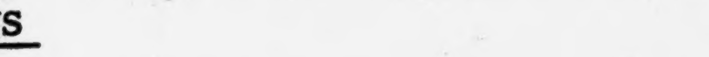
ORDER RELATIONS

Order relations are important in mathematics. They are now being introduced at the primary level and their use is extended to all grade levels. Traditional courses have stressed equalities but have given little or no attention to inequalities.

A number may be equal to, greater than, or less than a given number. These order relations are symbolized by = , > , and < . The symbol ">" is read "greater than" and the symbol "< " is read "less than". "6 > 4" is read "6 is greater than 4" and "3 < 4" is read "3 is less than 4". A number line may be used to illustrate these number relations.

NUMBER LINE

To prepare a number line draw a line and mark a sequence of equally spaced dots on it. Write '0' above the first dot, '1' above the second, '2' above the third, etc.



The farther a numeral is to the right, the greater the number it represents. For example, 7 > 2. The use of number lines and number line activities will be discussed in later units.

SUMMARY

Cardinal number--a natural number used to tell how many objects are in a set or collection, as three, five, etc., with no order implied.

Digits of the Hindu-Arabic decimal system--0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Natural numbers---0, 1, 2, 3, 4,...

Numeral---a written symbol used to represent a number.

One-to-one correspondence---a simple matching of one object to another or a simple pairing of objects.

Ordinal number--a natural number used to specify a particular element of an ordered set or collection, as page 82.

System of numeration---a system dealing with symbolism and ways of representing numbers.

SYMBOLS

$=$ is equal to

$\neq$ is not equal to

$>$ is greater than

$\geqslant$ is greater than or equal to

$\not>$ is not greater than

$\not\geqslant$ is not greater than or equal to

$<$ is less than

$\leqslant$ is less than or equal to

$\not<$ is not less than

$\not\leqslant$ is not less than or equal to

UNIT I

REFERENCES FOR TEACHERS

Adler--Exploring the World of Numbers and Space: pp. 9-13, 66

Adler--Mathematics: the Story of Numbers, Symbols, and Space: p. 4-7

Adler--The New Mathematics: pp. 13-16

Banks--Elements of Mathematics: pp. 5-6, 47-60, 79-84

C-2 Bulletin, Thinking In the Language of Mathematics: pp. 3-6, 27-42

Dantzig--Number, The Language of Science: pp. 1-12, 20-36

Hogben--The Wonderful World of Mathematics: pp. 6-11, 20-25

Larsen--Number Systems: pp. 3-15

NCTM, 24th Yearbook--The Growth of Mathematical Ideas: pp. 7-19, 327-337

NCTM, 25th Yearbook--Instruction in Arithmetic: pp. 1 - 47

Osborn, DeVault, Boyd, and Houston--Extending Mathematics Understanding: pp. 1-11, 129-137

Smith and Ginsburg--Numbers and Numerals: pp. 1-24

Swain--Understanding Arithmetic: pp. 1-31

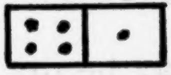
Titchmarsh--Mathematics for the General Reader: pp. 13-19, 23-25

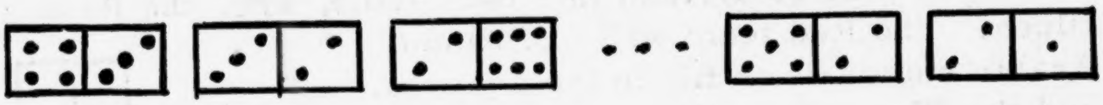
1.01 Flannel Board or Magnetic Board

Directions for constructing a flannel board are given on p. 142 of the C-2 Guide. If a sheet of tin plate is placed between the wood and the felt, the board can serve also as a magnetic board. Braces and mathematical symbols and numerals may be cut from felt for use on the flannel board. A symbol or numeral may be cut from cardboard and a small magnet glued to its reverse side so that it will stick to the flannel board.

1.02 The Missing Domino

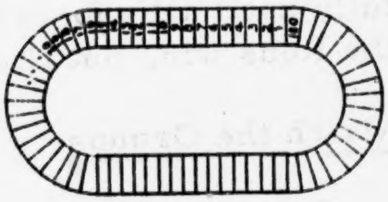
Withhold one domino. The rest may be matched and laid end-to-end around a table to provide practice in recognition of number sets. The teacher can identify the withheld domino if a student tells her the number of dots at each end of the completed domino row. These sets will match those on the missing domino. The missing domino for the train of dominoes pictured below would be:





1.03 Going to Town

Any student may place his marker at any space along the path but must return it to the same space by going all the way around. The spaces are numbered from 1-100. A spin of the top determines how many moves are to be made. For practice in learning the names and order of the numbers from 1-100, a student may count aloud as he moves his marker. This diagram may also be marked on a floor or play area.



GOING TO TOWN BOARD

TOP

1.04 Number Lines or Scales

Directions for constructing a number line are given on p. 139 of the C-2 Bulletin. An alternative to tag board is adding machine paper. Use larger colored numerals for 10, 20, 30, 40, 50, 60, 70, 80, 90, and 100. A number line may be constructed in different ways for different age groups. A number line for very young children may be made of spools, large beads, or large colored circles or squares on strips of paper. A number line constructed on the floor may be used for number line games. If the numerals are spaced a foot apart, the line may be used to develop measurement concepts. A number line may be extended to represent negative integers, or it may be subdivided to represent fractions and decimals.

1.05 A Hundred Board

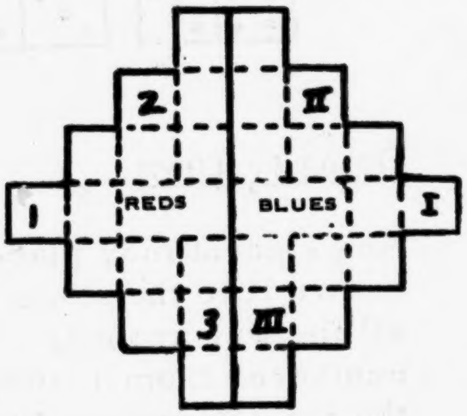
A Hundred Board may be constructed using a 24" x 30" piece of plyboard. The surface may be divided into 100 rectangular areas or a sheet of cardboard may be divided similarly and glued to the plywood. Colored cardboard has the advantage of being seen more easily and does not necessitate painting the plywood. The numerals 1-100 or 0-99 should be marked in each box. Hooks or screws may also be placed above each numeral. Small rings or paper cutouts can be hung on each screw or hook to point out patterns such as 5, 10, 15, 20, etc. or 4, 8, 12, 16, etc.

1.06 Calendars

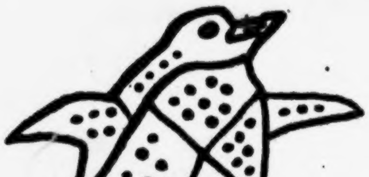
Attractive calendars can be made using numerals from an ancient system of numeration. This activity may be correlated with a project in social studies.

1.07 Follow the Pattern

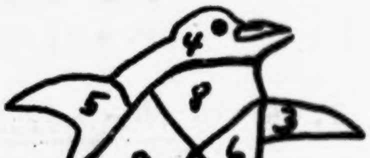
The class may be divided into two teams, e.g. the Reds and the Blues. The Red team will use Hindu-Arabic numerals to fill in the squares, and the Blue team will use Roman numerals. The Blue team may place a Roman numeral in any square on the blue side of the design. The Red team must place the corresponding Hindu-Arabic numeral in the corresponding square on the red side of the design. If this is done successfully until all squares are filled, the Reds win, but if a mistake is made the Blues win.



1.08 Match the Groups


MR. PENGUIN

Draw a picture of an animal or a bird on a large piece of oak tag or colored cardboard. Section the animal or the bird into ten spaces. In each space place a number of dots to represent the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.



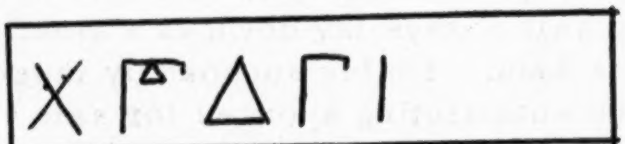
MATCH THE DOTS

Cut apart another animal or bird in identical sections. On the corresponding section write '1', '2', '3', etc. Let the students place these cutout sections on the dots which they describe. This can either be seatwork or an activity for the entire class if the picture of the animal or the bird is a large one.

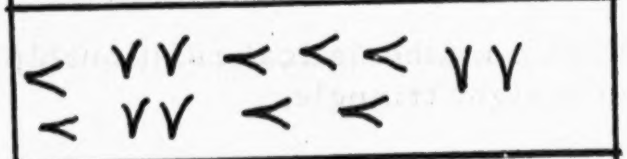
1.09 Famous Dates in History

Below are recorded years in which famous events in history occurred. Can you decipher the symbols used in these ancient systems of numeration and identify the ancient systems? Students may wish to find out what event in history is associated with each date.

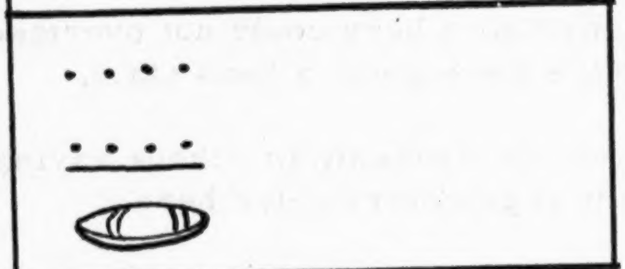
a.



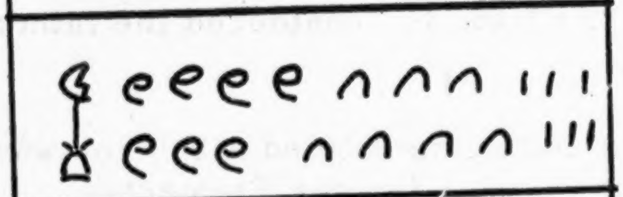
b.



c.

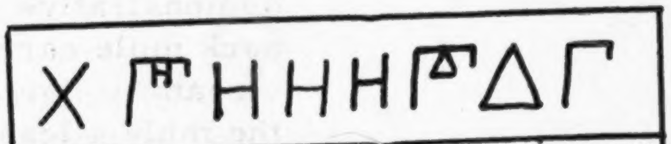


d.

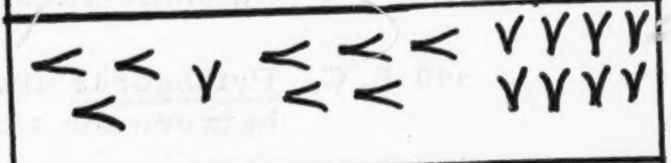


e. M D C C C X II

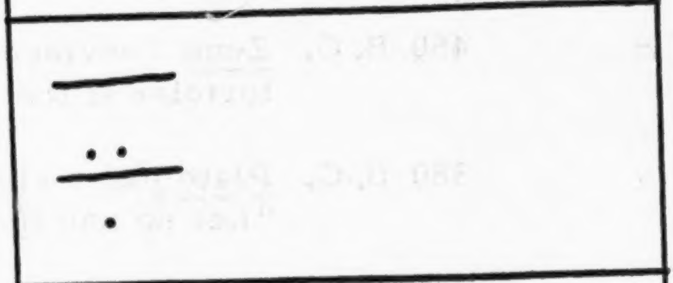
f.



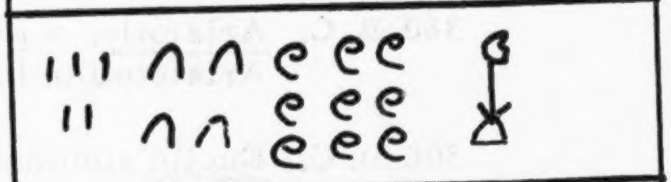
g.



h.



i.



j. MCMLXII

Answers:

(a) 1066 (Greek)
(b) 1492 (Babylonian)
(c) 1620 (Mayan)
(d) 1776 (Egyptian)
(e) 1812 (Roman)

(f) 1865 (Greek)
(g) 1918 (Babylonian)
(h) 1941 (Mayan)
(i) 1945 (Egyptian)
(j) 1962 (Roman)

1.10 Famous Mathematicians

Some students may become interested in reading about famous mathematicians. The list given here will be followed by an additional list in each succeeding unit. The mathematicians listed will be in chronological order with a brief note about each. An interesting activity for a student is to construct a mathematical time line from the completed list. A short biography and a picture of each mathematician may be put on each page of a notebook. Picture cards available from Scripta Mathematica, 186th and Amsterdam Avenue, New York 33, N. Y., may interest students in reading about some of these men.

600 B. C. Thales, one of the Seven Wise Men, began the study of demonstrative geometry. The story is told that his old pack mule carrying salt always lay down in a small stream to lighten its load. Thales supposedly taught the mule a lesson by substituting sponges for salt.

540 B. C. Pythagoras discovered a mathematical relationship between the sides of a right triangle.

450 B. C. Zeno convinced many that a hare could not overtake a tortoise if the tortoise were given a head start.

380 B. C. Plato had a sign over his Academy in Athens saying, "Let no one ignorant of geometry enter here".

360 B. C. Aristotle, a pupil of Plato, is considered the father of Aristotelian logic.

300 B. C. Euclid summarized and systematized all the mathematical knowledge of his time in his famous Elements.

250 B. C. Eratosthenes measured the circumference of the earth and calculated the distance from the earth to the sun and to the moon.

240 B. C. Archimedes is often considered one of the three greatest mathematicians of all time. An interesting story is recorded explaining how he determined that King Hiero's crown was not pure gold.

225 B. C. Apollonius summarized the known work on conics.

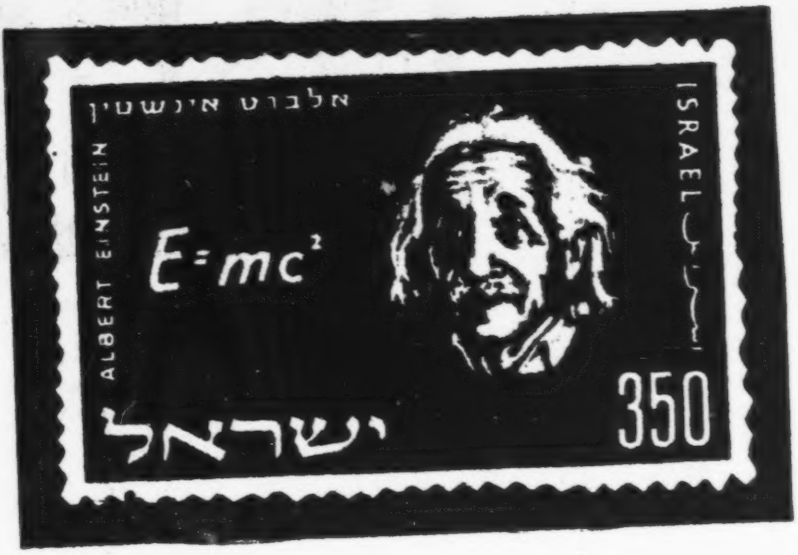
140 B. C. Hipparchus is often called the father of trigonometry.

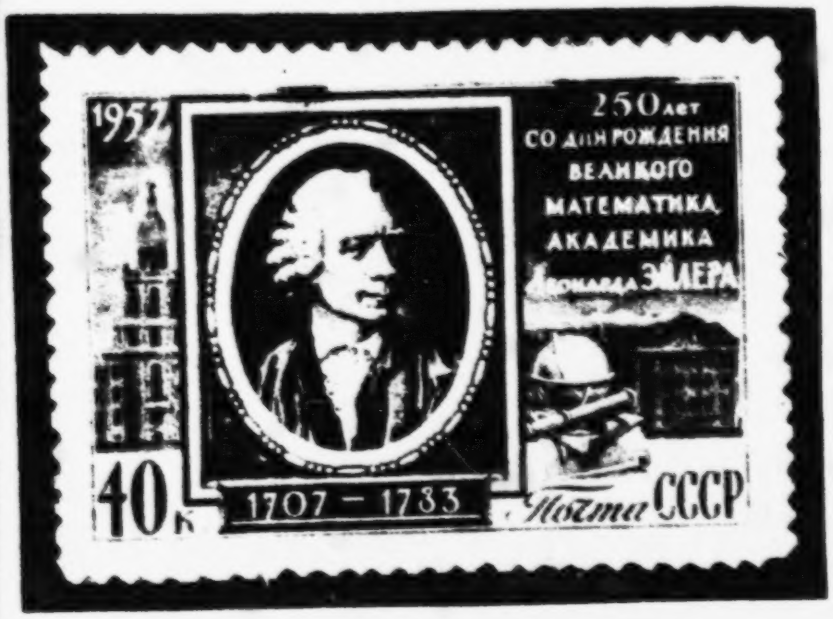
1.11 Stamps

Collecting stamps picturing famous mathematicians may interest some students. A short account of each mathematician with his picture may be put on each page of a notebook or the stamps can be kept in a stamp album. A short list of stamps is given below: (Country listed is country issuing.)

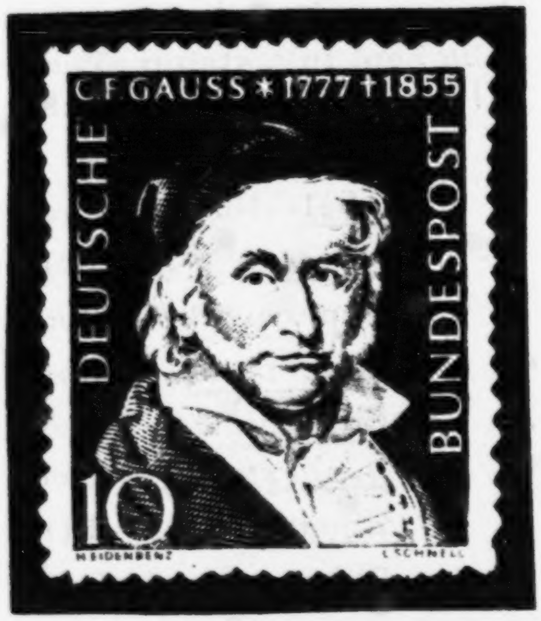
Abel--Norway (the stamp catalog number is A 18)
Boscovich--Croatia (1943 issue)
Copernicus--France (8-franc stamp of 1957 series)
Copernicus--Poland (2 stamps issued in 1953)
Copernicus--Russia (1955 issue)
Descartes--France (stamp catalog number is A 78)
Einstein--Israel (1956 issue)
Euler--Russia (1957 issue)
Galileo--Italy (4 stamps in 1942 series)
Gauss--West Germany (1955 issue)
Gerbert--Hungary (2 stamps issued in 1938)
Hamilton--Irish Free State (Catalog number is A 17)
Huygens--Netherlands (catalog number is SP 34)
Kepler--Austria (1953 issue)
Lagrange--France (1958 issue)
Laplace--France (1958 issue)
Leibniz--Germany (catalog number is A 57)
Leverrier--France (1958 issue)
Newton--France (18 franc stamp) 1957
Poincaré--France (1952 issue)
Pythagorean Theorem--Greece (one of 4 stamps in series)
Zeno--Cyprus (stamp number in catalog is A 16)

ALBERT EINSTEIN (1879-1955) IS REGARDED BY MANY AS THE GREATEST THEORETICAL PHYSICIST SINCE NEWTON. THIS STAMP WAS ISSUED IN 1956 BY ISRAEL SHORTLY AFTER HIS DEATH IN 1955. IT IS A 350-PRUTA AND IS DARK BROWN IN COLOR.

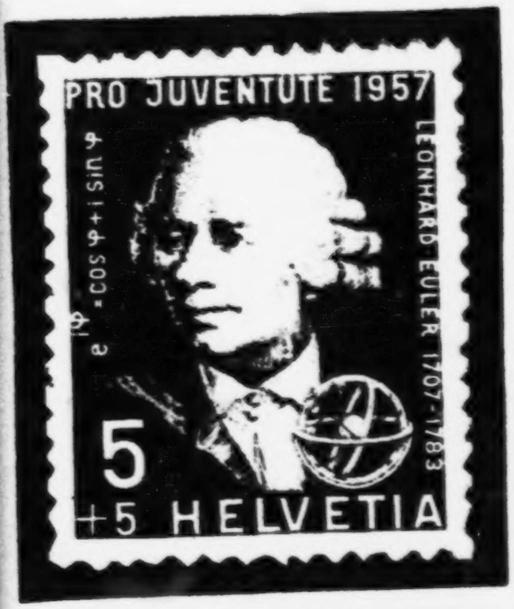



1957
250 лет
со дня рождения
великого
математика,
академика
Леонарда Эйлера
40к
1707 – 1783
Почта СССР

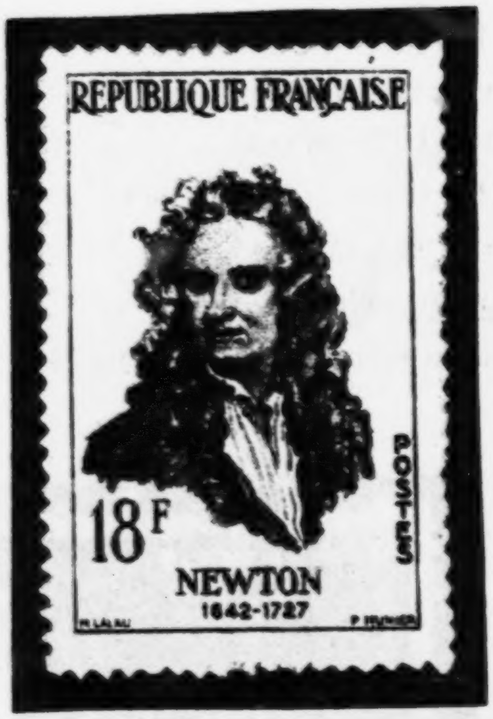
LEONHARD EULER, SWISS MATHEMATICIAN, (1707-1783) WAS ALSO HONORED BY RUSSIA IN 1957. THE STAMP PICTURED HAS A LILAC BACKGROUND WITH A GRAY INSET AND IS WORTH 40 KOPECKS.


C.F. GAUSS ✱ 1777 † 1855
DEUTSCHE
BUNDESPOST
10

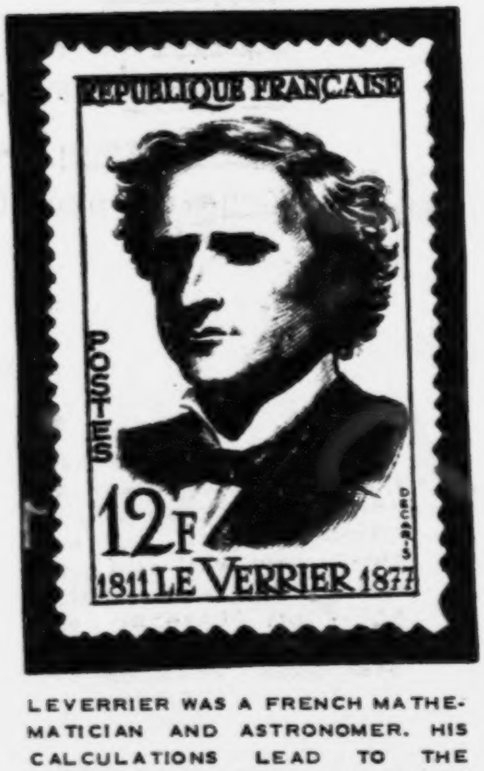
CARL FRIEDRICH GAUSS (1777-1855) IS PICTURED ON THIS DEEP GREEN 10-PFENNIG ISSUE OF WEST GERMANY. GAUSS IS OFTEN REFERRED TO AS "THE PRINCE OF MATHEMATICIANS." THIS OUTSTANDING GERMAN MATHEMATICIAN AND ASTRONOMER WAS EARLY RECOGNIZED AS A GENIUS.


PRO JUVENTUTE 1957
LEONHARD EULER 1707-1783
e^{iφ} = cos φ + i sin φ
5
+5 HELVETIA

IN 1957, SWITZERLAND PRINTED 5 + 5-CENTIME BROWN-CARMINE SEMIPOSTAL FOR THE 250TH BIRTHDAY ANNIVERSARY OF THE BRILLIANT ASTRONOMER AND MATHEMATICIAN, LEONHARD EULER


REPUBLIQUE FRANÇAISE
18F
NEWTON
1642-1727
POSTES

SIR ISAAC NEWTON IS CONSIDERED ONE OF THE GREATEST MATHEMATICIANS OF ALL TIME. HE WAS ALSO INTERESTED IN ASTRONOMY AND INVENTED THE NEWTONIAN REFLECTING TELESCOPE.


REPUBLIQUE FRANÇAISE
POSTES
12F
1811 LE VERRIER 1877

LEVERRIER WAS A FRENCH MATHEMATICIAN AND ASTRONOMER. HIS CALCULATIONS LEAD TO THE DISCOVERY OF NEPTUNE. PICTURED IS A 12-FRANC STAMP WHICH IS SEPIA AND GRAY IN COLOR. IT WAS ISSUED IN 1958 BY FRANCE.

EXERCISES

I.

Answer the following:

1. What is 1/2 of 8?
2. What is 1/2 of '8'?
3. Does '45' consist of 2 numbers?
4. Can numbers be found on this sheet?
5. Can 'numbers' be found on this sheet?
6. Can you erase the '4' in 24?

II.

Identify the following as illustrations of cardinal or ordinal number:

1) Edward VI

2) 10-word slogan

3) Four-leaf clover

4) Second prize

III.

Translate the following Roman numerals to Hindu Arabic notation:

1. XLIV
2. DCIX

IV.

Write using mathematical symbols:

1. 60 is greater than 20
2. 40 is less than y
3. a equals b
4. a is not equal to b
5. the set whose elements are a, b, and c

EXERCISES (answers) **UNIT I**

I.

1. 4
2. '3'or 'Ɛ,' etc.
3. No. A symbol or numeral can't consist of 2 numbers.
4. No. Numbers are ideas.
5. Yes. The word 'numbers' appears on this page.
6. No. A symbol can't be erased in a number because the number isn't made up of symbols.

II.

1. ordinal (tells which one)
2. cardinal (tells how many)
3. cardinal (tells how many)
4. ordinal (tells which one)

III.

1. 44

2. 609

IV.

1. $60 > 20$

2. $40 < y$

3. $a = b$

4. $a \neq b$

5. $\{a, b, c\}$

ASSIGNMENT

1. Answer the following yes or no:

   a. Is 4 > 6?

   b. Can you erase the '4' in 24?

   c. Is '9' smaller than '6' ?

   d. Does '46' consist of 2 numbers?

   e. Does '35' consist of a '3' and a '5'?

   f. Does '4' plus '1' equal '5'?

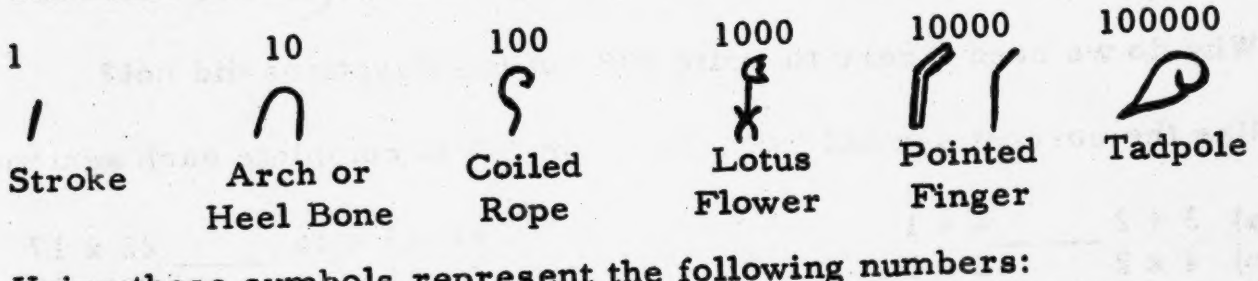
   g. Does 4 plus 1 equal 5?

   h. Do we write 1 first in writing '16'?

2. Identify the following as illustration of cardinal or ordinal numbers.

   (a) Henry VIII
   (b) page 82
   (c) 25 peaches
   (d) Rank 5 in class
   (e) 40-hour week
   (f) February 14
   (g) Fifth Symphony
   (h) License number 246-821
   (i) 1000 word theme
   (j) First prize

3. In Egyptian symbolism pictures of ropes, flowers, bones, etc. represented various numbers. The arrangement of the symbols did not matter and the flowers and ropes could turn either way. The most commonly used symbols included the following:

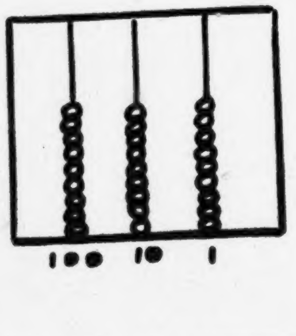

1
Stroke
10
Arch or Heel Bone
100
Coiled Rope
1000
Lotus Flower
10000
Pointed Finger
100000
Tadpole

   Using these symbols represent the following numbers:

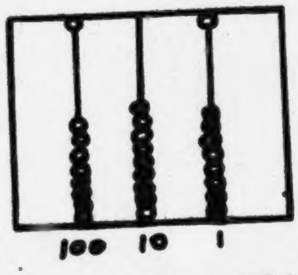
   a) 1 62
   b) 743
   c) 2146
   d) 91246
   e) 102,132

4. Make a simple sketch to show the bead arrangement on a nine-bead abacus of (a) 162 (b) 728 and (c) 4126

if



represents the abacus in zero state and



represents 101

5. Translate the following Roman numerals to their equivalents in the Hindu-Arabic notation. A bar over a letter or letters multiplies the value of the letter or letters by 1000.

   a) XXXVII
   b) XLVIII
   c) CXCI
   d) $\overline{\text{M}}$
   e) MCXCIV
   f) $\overline{\text{DC}}$II

6. Represent in Roman notation the numbers represented in Hindu-Arabic notation below:

   a) 462
   b) 814
   c) 500, 426
   d) 1961
   e) 1492
   f) 1650

7. Try solving without changing to Hindu-Arabic numerals but use Hindu-Arabic numerals if the problem becomes time consuming. This type of problem was solved by the Romans by the use of the abacus.

   a) MDVI - CCLXIV
   b) MCDVII + CCCXXIV
   c) (XLIX) x (VI)
   d) (XLII) ÷ (VI)

8. Which of the following principles--repetition, addition and place value apply to:

   a) Hindu-Arabic system?
   b) Egyptian system?
   c) Babylonian system?
   d) Mayan system?
   e) Roman system?

9. Why do we need a zero to write 408 but the Egyptians did not?

10. Use the correct symbol "<", ">", or "=" to complete each sentence.

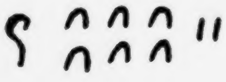
    a) 3 + 2 _____ 4 + 1
    b) 4 x 2 _____ 3 x 3
    c) 3 + 6 - 2 _____ 3 x 2
    d) 14 - 10 + 7 _____ 6 + 7 - 4 + 2
    e) 23 x 16 _____ 22 x 17
    f) 32 x 73 _____ 73 x 32
    g) 181 + 19 _____ 19 + 181
    h) 6 x 7 x 4 _____ 4 x 6 x 7

ASSIGNMENT (answers)

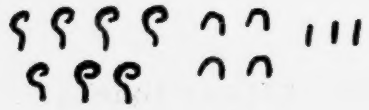
1. a. No. The idea of 4 is less than the idea of 6.
   b. No. You cannot erase part of a number idea.
   c. Yes. The numeral '9' is smaller than the numeral '6'.
   d. No. Numerals don't consist of numbers.
   e. Yes. The numeral '35' is written using the numerals '3' and '5'.
   f. No. We don't add numerals. We add numbers.
   g. Yes.
   h. No. We can't write numbers. We write the numeral '1' first.

2. a. Ordinal (which one)
   b. Ordinal (which one)
   c. Cardinal (how many)
   d. Ordinal (which one)
   e. Cardinal (how many)
   f. Ordinal (which one)
   g. Ordinal (which one)
   h. Probably ordinal since licenses are numbered serially (which one)
   i. Cardinal (how many)
   j. Ordinal (which one)

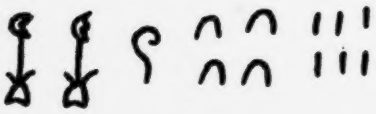
3. a



b.



c.



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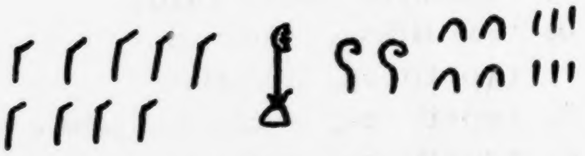
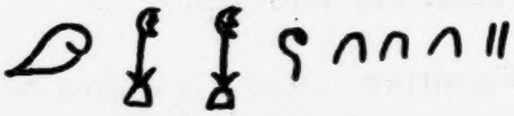
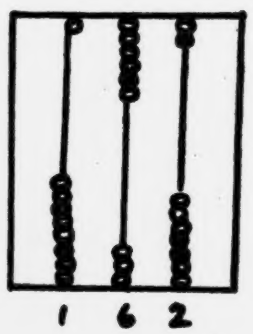


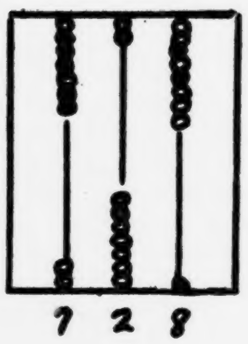
e.



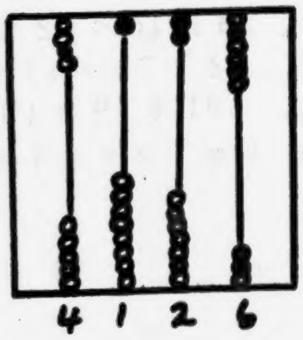
4. a.



b.



c.



5. a. 37
   b. 48
   c. 191
   d. 1,000,000
   e. 1194
   f. 600,002

ASSIGNMENT (answers)

6. a. CDLXII c. $\overline{\text{D}}$CDXXVI e. MCDXCII
   b. DCCCXIV d. MCMLXI f. MDCL

7. a. MDVI - CCLXIV. Many ways, but probably change as follows:
   MCCCCLXXXXXIIIIII - CCLXIIII = MCCXXXXII or MCCXLII.
   Remember they did this on an abacus. Check: 1506-264 = 1242.

   b. MCDVII + CCCXXIV = MCDCCCXX I VV I I = MDCCXXXI
   Check: 1407 + 324 = 1731

   c. (XLIX) x (VI) actually shows repeated addition or XXXXVIIIIXXXXVIII
   etc. = XXXXXXXXXXXXXXXXXXXXXXXXVVVVVVIIIIIIIIIIIIIIIIIIIIIIII =
   CCLXXXXIV or CCXCIV. (49) • (6) = 294.

   d. XLII ÷ VI
   XLII = VVVVVVVVII = VVVVVVVIIIIIII which represents VII
   groups of VI. 42 ÷ 6 = 7

8. a. addition, place value
   b. repetition, addition
   c. repetition, addition
   d. repetition, addition, place value
   e. repetition, addition

9. The Egyptians used a repetitive and additive system rather than a positional and additive system.

10. a. $3 + 2 = 4 + 1$
    b. $4 \times 2 < 3 \times 3$
    c. $3 + 6 - 2 > 3 \times 2$
    d. $14 - 10 + 7 = 6 + 7 - 4 + 2$
    e. $23 \times 16 < 22 \times 17$
    f. $32 \times 73 = 73 \times 32$
    g. $181 + 19 = 19 + 181$
    h. $6 \times 7 \times 4 = 4 \times 6 \times 7$

UNIT II

SYSTEMS OF NUMERATION

UNIT II. . SYSTEMS OF NUMERATION

I'm glad I'm educated
I think it's simply grand
To know so many facts and stuff
That I don't understand.

-----Barker

INTRODUCTION

"1 + 1 = 10" is only a way of saying that "1 + 1 = 2". This usage of "10" does not indicate ten but indicates "one two and zero ones". Systems of numeration in bases other than ten are frequently included in modern courses in arithmetic to create interest and to help clarify the decimal system of numeration.

DECIMAL SYSTEM

The Hindu-Arabic system, which is used in most of the civilized world, is a decimal system based on groupings of ten. The word "decimal" is derived from the Latin "decem" meaning "ten". For this reason the system is often called the base ten system. Decimal or base ten systems use only ten basic symbols called digits.

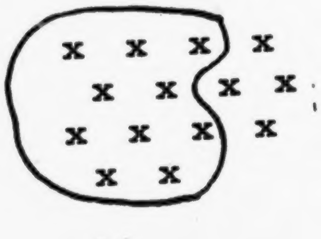
An important feature of the Hindu-Arabic additive decimal system is the use of positional or place value which makes it possible to represent a number of any magnitude using only the ten basic symbols (digits)-- 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The position of each digit in the numeral determines the value of that digit. For example, in '56' the '5' denotes 5 tens while in '546' the '5' denotes 5 hundreds.

This idea of position changed numerical notation and existing algorithms (computational patterns). Before the invention of a positional system, mathematics was handicapped in its advancement because calculations in the older systems were too cumbersome.

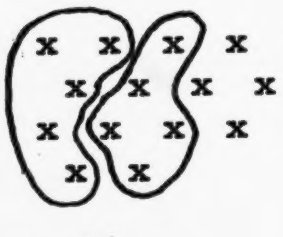
BASE

The base of any system of numeration indicates the number of single objects in the basic group and the number of digits needed. In the decimal system, 14 indicates one group of ten and four ones.

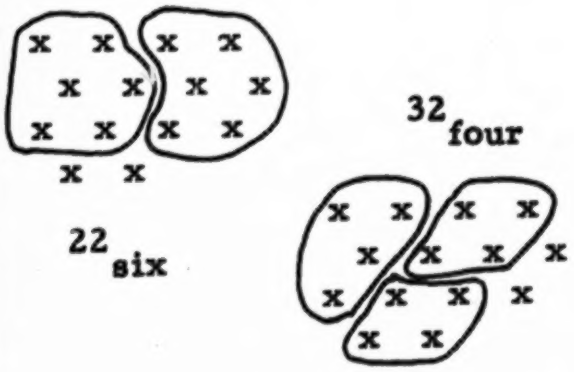
Objects may be grouped in other ways. If the objects at the left were grouped in sets of five, there would be 2 sets of five and 4 ones. This grouping could be recorded as 24_{five} or 24_5 and read "two four, base five" or "2 five 4".



14_{ten}



24_{five}



If these objects were grouped in sets of six, there would be 2 sets of six and 2 ones. This grouping could be recorded as 22_{six} or 22_6 and read "two two, base six" or "2 six 2".

If these objects were grouped in sets of four, there would be 3 sets of four and 2 ones. This grouping could be recorded as 32_{four} or 32_4 and read "three two, base four" or "3 four 2".

The four numerals--14_{10}, 24_5, 22_6, and 32_4--indicate the same number of objects but the bases indicate different groupings. In each numeral the digit at the right indicates the ones and the next digit to the left indicates the number of basic groups.

Strictly speaking, the symbol "10" read "one zero" always indicates the base of a system of numeration since it indicates one basic group. Thus it represents ten only when the base is ten. It would represent two if the base were two, or it would represent six if the base were six.

EXPONENTS AND POWERS

In addition to the characteristics of base and place value in an additive positional system of numeration, a third characteristic is the exponential aspect.

The number 1000 may be represented as '10^3'. The expression "10^3" is read "ten to the third power". Ten is called the base, 1000 or 10^3 is called the power, and the small superscript 3 is called the exponent. By definition the exponent, if a natural number, indicates the number of times the base is used as a multiplier or factor. Thus $8^4 = 8 \times 8 \times 8 \times 8$ and $B^2 = B \cdot B$.

$$B^3 \cdot B^2 = (B \cdot B \cdot B) \cdot (B \cdot B) = B^5 \text{ and } \frac{B^5}{B^2} = \frac{B \cdot B \cdot B \cdot B \cdot B}{B \cdot B} = B^3$$

It may be observed that a short way to multiply powers of the same base is to add the exponents. To divide powers of the same base subtract the exponents.

Note that $\frac{B^3}{B^3} = \frac{B \cdot B \cdot B}{B \cdot B \cdot B} = 1$

However, using the shorter method above:

$$\frac{B^3}{B^3} = B^{3-3} = B^0$$

To have the results consistent, the following definition has been agreed upon:

$$B^0 = 1 \qquad \text{if } B \neq 0$$

A smaller power may be divided by a larger power:

$$\frac{B^2}{B^3} = \frac{B \cdot B}{B \cdot B \cdot B} = \frac{1}{B}$$

Again the shorter method may be used:

$$\frac{B^2}{B^3} = B^{2-3} = B^{-1}$$

This indicates a need for the definition:

$$B^{-1} = \frac{1}{B}$$

Similarly, it can be shown that

$$\frac{B^2}{B^4} = B^{-2} = \frac{1}{B^2}$$

EXPONENTIAL NOTATION IN BASE TEN

When the base is given, a number can be expressed in exponential form. Another way of writing 5246 is to use exponential notation to indicate the groupings:

$$\begin{aligned} 5246 &= 5000 + 200 + 40 + 6 \\ &= 5(1000) + 2(100) + 4(10) + 6(1) \\ &= 5(10^3) + 2(10^2) + 4(10^1) + 6(10^0) \end{aligned}$$

Groupings in base ten from 10^3 to 10^{-2} are represented in the following table:

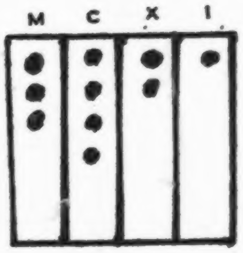
thousand	hundred	ten	one	tenth	hundredth
1000	100	10	1	1/10	1/100
$10(10^2)$	$10(10^1)$	$10(10^0)$	$10(10^{-1})$	$10(10^{-2})$	$10(10^{-3})$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}

Below is shown the composition of 210.45:

$$\begin{array}{rll} 200. & = 2 \times 100 & = 2 \cdot 10^2 \\ 10. & = 1 \times 10 & = 1 \cdot 10^1 \\ 0. & = 0 \times 1 & = 0 \cdot 10^0 \\ .4 & = 4 \times 1/10 & = 4 \cdot 10^{-1} \\ .05 & = 5 \times 1/100 & = 5 \cdot 10^{-2} \\ \hline 210.45 & \multicolumn{2}{l}{2\ 10^2 + 1 \cdot 10^1 + 0 \cdot 10^0 + 4 \cdot 10^{-1} + 5 \cdot 10^{-2}} \end{array}$$

USE OF OTHER BASES

Bases other then ten were used in some of the ancient systems of numeration. For example, the ancient Babylonians used a sexagesimal system (base 60), the Mayas in Yucatan had a modified system that was primarily vigesimal (base 20), and various tribes in Africa and Australia used bases of 2, 3, or 4.



ROMAN ABACUS

Although the base notation may not be specified, the idea of other bases is frequently used today in various kinds of measurements. In measuring time and angles, the idea of a sexagesimal base is used.
For time: 60 seconds = 1 minute, 60 minutes = 1 hour.
For angles: 60 seconds = 1 minute, 60 minutes = 1 degree.

To measure collections of articles by the dozen, gross and great gross, or to measure the number of months in a year, or to measure the number of inches in a foot, the idea of a duodecimal base is used.
12 articles = 1 dozen; 12 dozen = 1 gross; 12 gross = 1 great gross.

The idea of base eight is used by stockbrokers in quoting prices in eighths. Base two, called the binary or dyadic base, finds practical application in electronic computers. Also base two is used in an everyday measure: 2 cups = 1 pint; 2 pints = 1 quart; 2 quarts = 1 half gallon; 2 half gallons = 1 gallon.

The idea of base five is shown by: 5 cents = 1 nickel; 5 nickels = 1 quarter.

QUINARY SYSTEM

A quinary (base five) system of numeration uses groupings of five and requires five symbols including the symbol for zero. A number line in base five is easily constructed using the symbols 0, 1, 2, 3, and 4. Note that 10 follows 4.

0 1 2 3 4 10 11 12 13 14 20 21 22 23 24 30 ...

Use this number line to verify the addition and multiplication tables shown below:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Use the addition and multiplication tables to verify the following examples written in base five symbolism:

a) $$\begin{array}{r} 41 \\ +\ 23 \\ \hline 114 \end{array}$$

b) $$\begin{array}{r} 32 \\ 43 \\ +\ 34 \\ \hline 214 \end{array}$$

c) $$\begin{array}{r} 42 \\ -\ 13 \\ \hline 24 \end{array}$$

d) $$\begin{array}{r} 34 \\ \times\ 13 \\ \hline 212 \\ 34 \\ \hline 1102 \end{array}$$

Perform the indicated operations below and compare your results with those given on page 28.

1) $$\begin{array}{r} 24 \\ +\ 23 \\ \hline \end{array}$$

2) $$\begin{array}{r} 42 \\ 13 \\ +\ 24 \\ \hline \end{array}$$

3) $$\begin{array}{r} 32 \\ \times\ 24 \\ \hline \end{array}$$

4) $$\begin{array}{r} 141 \\ \times\ 23 \\ \hline \end{array}$$

BINARY SYSTEM

Many electronic computers use the binary system because of its simplicity. Since only the symbols 0 and 1 are needed to write any numeral in base 2, computation in this system is interesting. The following table shows the comparison between base 10 and base 2 numerals:

BASE 10	BASE 2
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101

BASE 10	BASE 2
14	1110
15	1111
16	10000
17	10001
18	10010
19	10011
20	10100
21	10101
22	10110
23	10111
24	11000
25	11001
26	11010

Although more space is required to write a numeral in base 2, the system has the advantage of few addition and multiplication facts. Tables for these facts are given below:

+	0	1
0	0	1
1	1	10

x	0	1
0	0	0
1	0	1

Note that: 1 + 1 = 10 and 10 - 1 = 1

1 + 1 + 1 = 11 11 - 1 = 10

1 + 1 + 1 + 1 = 100 100 - 1 = 11

Verify that the computation for each of the following examples in base two is correct:

Add:

$$\begin{array}{r} 1111 \\ 101 \\ \hline 10100 \end{array}$$

Subtract:

$$\begin{array}{r} 1100 \\ 101 \\ \hline 111 \end{array}$$

Multiply:

$$\begin{array}{r} 110 \\ 11 \\ \hline 110 \\ 110 \\ \hline 10010 \end{array}$$

Divide:

$$\begin{array}{r} 110 \\ 10\,\overline{)\,1101} \\ 10 \\ \hline 10 \\ 10 \\ \hline 1 \end{array}$$

Use base two notation and follow the directions given below. Compare your results with those given on page 28.

5. Add:

$$\begin{array}{r} 11011 \\ 1101 \\ \hline \end{array}$$

6. Subtract:

$$\begin{array}{r} 111001 \\ 10110 \\ \hline \end{array}$$

7. Multiply:

$$\begin{array}{r} 1101 \\ 111 \\ \hline \end{array}$$

8. Divide : $101\,\overline{)\,1000001}$

CHANGING TO BASE TEN FROM ANOTHER BASE

Changing from one base to another is a problem of regrouping. Changing to base ten from another base is a problem of regrouping by ones, tens, hundreds, etc.

473_{eight} indicates $4\cdot10^2 + 7\cdot10^1 + 3\cdot10^0$ which in decimal notation equals $4\cdot8^2 + 7\cdot8^1 + 3\cdot8^0$ or $4\cdot64 + 7\cdot8 + 3\cdot1$ or 315.

Similarly, 231_5 may be thought of as $2 \cdot 5^2 + 3 \cdot 5^1 + 1 \cdot 5^0$ or $2 \cdot 25 + 3 \cdot 5 + 1$ or 66_{10}. There is, of course, no numeral '5' in base five since the base is represented by '10'. However, at this time it is convenient to use base ten numerals to avoid misunderstanding.

The use of a chart often aids in changing from some given base to base ten or vice versa. To change 110101_2 to base ten, the chart below may be used:

	2^5	2^4	2^3	2^2	2^1	2^0
Size of Group	32	16	8	4	2	1
Number of Groups	1	1	0	1	0	1

Then $110101_2 = 1 \cdot 32 + 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1$ or $32 + 16 + 4 + 1$ or 53_{10}.

CHANGING FROM BASE TEN TO ANOTHER BASE

Changing from base ten to a different base is again a problem of regrouping. In base ten notation 315 represents 3 groups of 100, 1 group of 10, and 5 ones. A regrouping of 315_{ten} using basic groups of eight is shown in the table:

Size of groups	8^2	8^1	8^0
Number of groups	4	7	3

Since 8^3 equals 512 and 8^2 equals 64, the largest power of 8 that is less than 315 is 64. 315 contains 4 groups of 64 with 59 remaining. 59 consists of 7 groups of 8 with 3 remaining. Thus 315_{10} is equivalent to 473_8

$$\begin{array}{rl} 315 & \\ -\ \underline{256} & \text{4 groups of 64} \\ 59 & \\ -\ \underline{56} & \text{7 groups of 8} \\ 3 & \text{3 ones} \end{array}$$

This same regrouping by eights and then by eights again can be accomplished by the algorithm below:

$$\begin{array}{l} 8 \,\underline{|\ 315\ } \\ \quad 8 \,\underline{|\ 39\ } - 3 \\ \qquad\quad 4 \ - 7 \end{array}$$

The first division or grouping by 8 indicates 39 groups of eight and 3 ones. The second division by 8 results in 4 groups of 8 eights with 7 groups of eight remaining. The numeral 473_8 is read directly from the algorithm.

Rewrite the numerals below in the indicated bases and check the results with those below:

9. $42_8 = \underline{\qquad\quad}_{10}$

10. $143_5 = \underline{\qquad\quad}_{10}$

11. $82_{10} = \underline{\qquad\quad}_{7}$

12. $146_9 = \underline{\qquad\quad}_{10}$

13. $28_{10} = \underline{\qquad\quad}_{2}$

14. $2153_6 = \underline{\qquad\quad}_{10}$

15. $124_{10} = \underline{\qquad\quad}_{3}$

16. $342_{10} = \underline{\qquad\quad}_{8}$

17. $73_{10} = \underline{\qquad\quad}_{2}$

18. $34.2_5 = \underline{\qquad\quad}_{10}$

ANSWERS

1) 102
2) 134
3) 1423
4) 4343
5) 101000
6) 100011
7) 1011011
8) 1101
9) 34
10) 48
11) 145
12) 123
13) 11100
14) 501
15) 11121
16) 526
17) 1001001
18) 19.4

REFERENCES FOR TEACHERS UNIT II

Adler--Magic House of Numbers: pp. 25-42, 90-106

Adler--Mathematics, Exploring the World of Numbers and Space: pp. 20-21

Banks--Elements of Mathematics: pp. 59-77

Dantzig--Number, The Language of Science: pp. 12-19

Johnson and Glenn--Computing Devices: pp. 15-22, 37-48

Johnson and Glenn--Understanding Numeration Systems: pp. 1-44

Larsen--Binary Numbers: pp. 1-15

Larsen--Excursions in Arithmetic: pp. 6-7

Larsen--Games to Play: pp. 4-7

Larsen--Ways to Multiply: pp. 10-13

Merrill--Excursions in Mathematics: pp. 13-22

Osborn, DeVault, Boyd, and Houston-Extending Mathematics Understandings: pp. 12-24, 138-155

Swain--Understanding Arithmetic: pp. 97-112, 217-218

ACTIVITIES

.01 Bachet's Weight Problem

Many puzzles are based on other systems of numeration. One based on the ternary system (base 3) is used in weight puzzles.

Usually the problem is to determine the fewest weights needed to weigh any integral number of pounds from 1 to 40 pounds inclusive. This is an old problem and is sometimes called Bachet's weight problem.

In this case only 1-pound, 3-pound, 9-pound, and 27-pound weights need to be used.

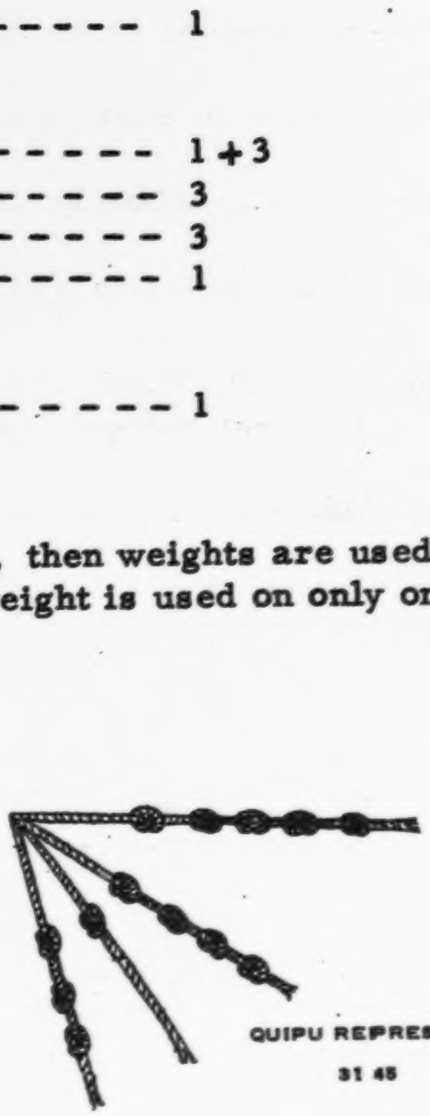
Write to 12 in the ternary system to determine how the weights are to be used for the first 12 pounds:

Decimal	Ternary	(L) Weighing (R)
1	1	1
2	2	3 ----- 1
3	10	3
4	11	3 + 1
5	12	9 ----- 1 + 3
6	20	9 ----- 3
7	21	9 + 1 ----- 3
8	22	9 ----- 1
9	100	9
10	101	9 + 1
11	102	9 + 3 ----- 1
12	110	9 + 3

If the ternary numeral includes a digit '2', then weights are used on both sides of the scale. If there is no '2', a weight is used on only one side.

.02 Quipu

A quipu may be used to represent numbers in various bases. Using cords of different colors, assign a place value to each color, e.g., red-ones, blue-tens, etc. To represent a number tie knots in the appropriate cords. The quipu picture represents 3,145. Assign different values to the cords to represent dollars and cents.



QUIPU REPRESENTING

31 45

ACTIVITIES

2.03 Puzzle-Base 2

A man had 1000 sheep. In the market-place there were 10 pens. How many sheep must the man put in each pen so that if a buyer asked for any number of sheep from 1 to 1000 the man could make the sale directly by emptying out some or all of the pens? The man is not allowed to take only some sheep from a pen; he must take all.

The puzzle above makes use of base two.

$2^0 = 1$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32$

$2^6 = 64$

$2^7 = 128$

$2^8 = 256$

Before this puzzle is presented to students, challenge them to determine how to find the sum of the first n numbers expressed as powers of 2:

$1 + 2 = 3$; $1 + 2 + 4 = 7$; $1 + 2 + 4 + 8 = 15$. Continue until students can generalize the result: $1 + 2 + 4 + \cdots 2^{n-1} = 2^n - 1$. This generalization can be used in the puzzle problem to determine the number of sheep in the first 9 pens.

The answer is shown below. Since the number of sheep in the first 9 pens is 511, the number in the last pen will be 489 instead of 2^9.

16	8	4	2	1
32	64	128	256	489

2.04 Inventing Names for Numbers

Children enjoy making up names for their own symbols in other bases. A sample system using base 4 is given below:

1	un
2	do
3	tre
10	tan
11	tanun
12	tando
13	tantre
20	dotan
21	dotanun
22	dotando
23	dotantre
30	tretan
31	tretanun
32	tretando
33	tretantre
100	han
101	hanun
102	hando
1000	man
10000	pan
	etc.

2.05 Inventing Symbols to Represent Numbers

Bright children like the opportunity to invent a system of numeration using their own symbols, names and bases. Activities that may be associated with such a project are writing the consecutive numerals in the base, constructing addition and multiplication tables, and solving problems.

A student may use such symbols as □ , I , ∧ and Δ to correspond to 0, 1, 2, and 3 respectively. The first twelve numerals are written as follows:

Base 10	Base I□	Base 10	Base I□
1	I	7	IΔ
2	∧	8	∧□
3	Δ	9	∧I
4	I□	10	∧∧
5	II	11	∧Δ
6	I∧	12	Δ□

.06 Nim (A Variation)

Nim is a game for two players. From a given pile each player picks up any number of toothpicks, say from 1 to 5. The maximum number to be picked up must be agreed upon. The play is alternate and whoever has to pick up the last toothpick is the loser.

Suppose the original pile contains 21 toothpicks and the maximum number to be picked up is 5. In that case, the one playing first can force a win by mentally dividing the toothpicks into groups, 1, 6, 6, 6, and 2. If he plays first, he picks up 2 toothpicks. Then regardless of how many his opponent picks up, the first player picks up enough to total 6. If the second player takes 1, the first player takes 5; if the second player takes 2, the first player takes 4, and so on. This exhausts the three groups of 6 and the second player is left with the last toothpick.

The game may be generalized by permitting a player to pick up any number of toothpicks from 1 to n. If the original pile contains x toothpicks, divide x by (n + 1). If you play first, pick up (r-1) toothpicks where r is the remainder when x is divided by (n + 1). After that, pick up enough toothpicks to total (n + 1).

ACTIVITIES

2.07 Napier's Rods

A project that many students will enjoy and one that provides them with an added opportunity to learn the multiplication facts as well as something of the historical development of mechanical devices for computing is the construction of a set of Napier's Bones or Rods.

Using tongue depressors, construct an index bone as illustrated and one or more bones for each set of multiplication facts for the numbers 0 to 9. The index bone and bones for 1, 2, 4, and 9 are illustrated.

To multiply 49 by 4, use the index, the four, and the nine bones as shown in Figure 2. Read the numerals on the line to the right of 4 on the index bone--196. The 6 and 3 are added since each indicates the number of tens.

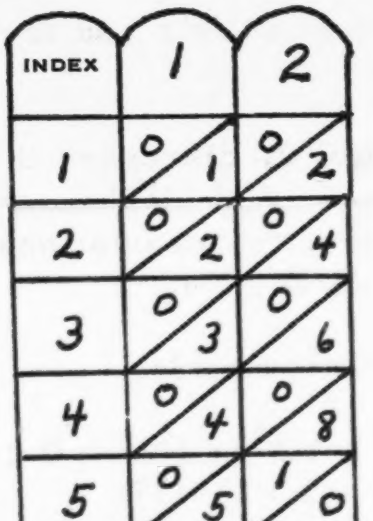


Fig. 1

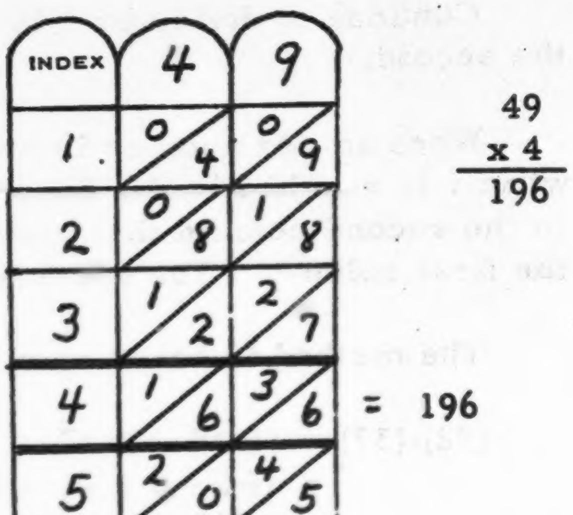


Fig. 2

NAPIER'S RODS

When the multiplier has more than two digits, then the partial products are found from the bones and written in the usual manner.

Constructing bones for a system other than base ten is an interesting project.

2.08 Binary System: Russian Peasant Method of Multiplication

The so-called "Russian Peasant Method of Multiplication" is based upon the binary system. There are said to be places in Africa where this method is still used.

92	x	37	
46		74	
*23		148	148
*11		296	296
* 5		592	592
2		1184	
* 1		2368	2368
			3404

To multiply 92 x 37, write '92' at the top of one column and '37' at the top of another.

Divide 92 by 2 and multiply 37 by 2, writing the results below the original numerals.

Continue to divide by 2 in the first column and to multiply by 2 in the second.

When an odd number is divided by 2, disregard the remainder. Stop when 1 is reached in the first column. Add all the numbers represented in the second column that are opposite representations of odd numbers in the first column. The correct result is 3404.

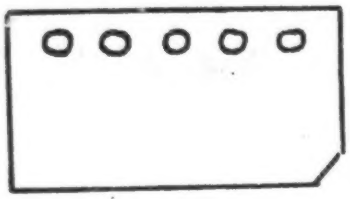
The method is based upon the binary scale.

$$\begin{aligned}(92)\cdot(37) &= (1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 0\cdot 2^0)\cdot(37)\\ &= (64 + 0 + 16 + 8 + 4 + 0 + 0)\cdot(37)\\ &= 2368 + 0 + 592 + 296 + 148 + 0 + 0\\ &= 3404\end{aligned}$$

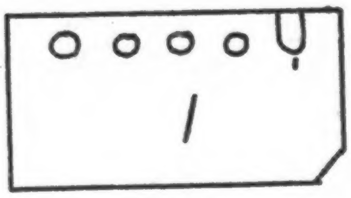
2.09 Card Sorting Using the Binary System

A simple sorting process can be shown using index cards and a nail or pencil as a sorter.

To make 32 cards, punch five holes across the top of 32 index cards as shown in A.

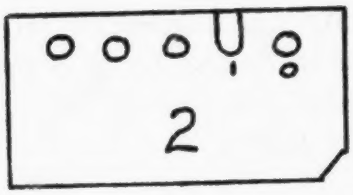


A

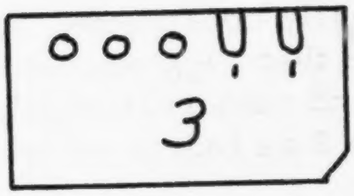


B

The card A represents the number 0. To represent 1, make a cut as in B. To represent 2 and 3, make cuts as shown in C and D respectively.



C



D

Make a card correspond to each numeral 0 through 31 in the binary system. The cuts correspond to the 1's in the numbers. Shuffle the cards. Then insert a small nail through the holes at the right. Shake the cards and place those that remain on the nail in front of the others. Next insert the nail in the second hole from the right. Again shake the cards. Place those that remain on the nail in front.

Continue this process until the nail has been inserted in each hole in order; each time place those cards remaining on the nail in front. After the fifth time the cards will be arranged in numerical order from 0 to 31.

If 6 holes are punched, the cards may be numbered from 0 to 63.

2.10 Binary System: Age Tables

Age tables or "magic" cards based upon the binary system are interesting. Ask a friend to select the cards on which his age appears. By adding the numbers in the upper left hand corners of all the cards he indicates, you can tell him his age.

1	3	5	7	9	11
13	15	17	19	21	23
25	27	29	31	33	35
37	39	41	43	45	47
49	51	53	55	57	59

2	3	6	7	10	11
14	15	18	19	22	23
26	27	30	31	34	35
38	39	42	43	46	47
50	51	54	55	58	59

4	5	6	7	12	13
14	15	20	21	22	23
28	29	30	31	36	37
38	39	44	45	46	47
52	53	54	55	60	

8	9	10	11	12	13
14	15	24	25	26	27
28	29	30	31	40	41
42	43	44	45	46	47
56	57	58	59	60	

16	17	18	19	20	21
22	23	24	25	26	27
28	29	30	31	48	49
50	51	52	53	54	55
56	57	58	59	60	

32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
56	57	58	59	60	

With six cards all the numbers up to, but not including 64, can be represented since $1 + 2 + 4 + 8 + 16 + 32 = 63$ or $2^6 - 1 = 63$. These tables are easily constructed by representing all numbers in the binary base. Notice that 14_{10} equals 1110_2 and it is found on the cards with upper left-hand numerals of '2', '4', and '8'. $14_{10} = 1110_2 = 2^3 + 2^2 + 2^1 = 8 + 4 + 2$ as indicated by $1\cdot2^3 + 1\cdot2^2 + 1\cdot2'$.

Notice that '14' is found on the cards whose upper left-hand numerals are '2', '4', and '8'. This can be explained by representing 14 in the binary base and examining the groups.

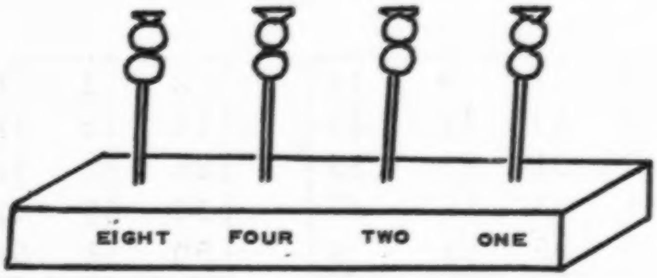
$14_{10} = 1110_2$ and $1110_2 = 1\cdot2^3 + 1\cdot2^2 + 1\cdot2^1 + 0\cdot2$. $2^3 = 8$, $2^2 = 4$, and $2^1 = 2$.

2.11 Duodecimal Base

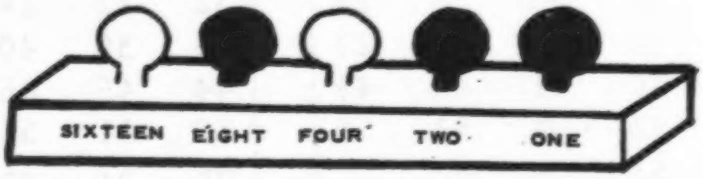
Find out about the base twelve (duodecimal) system of numeration which uses the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X and Ɛ. The last two symbols are read "dek", and "el"; 10 representing twelve is read "do". This system is advocated by the Duodecimal Society of America which publishes its own magazine with all computation done in base twelve.

.12 Abacus for Other Bases

An abacus for any base may be constructed using beads and nails or styrofoam balls and sticks. A small board may be used for the base. An abacus for base two is shown below:



A board using small flashlight bulbs or Christmas tree bulbs may be wired and used to represent numbers in the binary base. If the light is on, it represents '1'. If the light is off, it represents '0'. The lights below represent 11_{ten} or 1011_{two}.



2.13 Changing from Base N to Base N^3 (and vice versa)

It is easy to change directly from base N notation to base N^3 notation or vice versa and this finds practical application in electronic computers.

To change 110010111_2 to base eight notation, begin at the right and group the digits by 3's; change each numeral represented by each group of digits to the corresponding numeral in base eight notation.

$$\begin{tikzcd} 110_2 \arrow[d] & 010_2 \arrow[d] & 111_2 \arrow[d] \\ 6_8 & 2_8 & 7_8 \end{tikzcd}$$

'110010111_2' represents the same number as '627_8'.

To change from base N^3 notation to base N notation, represent each digit in the base N^3 numeral in base N notation. For example, to change 362_8 to base two notation, represent 3_8, 6_8, and 2_8 in base two notation:

$$\begin{tikzcd} 3_8 \arrow[d] & 6_8 \arrow[d] & 2_8 \arrow[d] \\ 11_2 & 110_2 & 010_2 \end{tikzcd} \quad \text{or } 362_8 = 11110010_2$$

2.14 Changing from Base N to Base N^2 Notation (and vice versa)

The same scheme may be used to change a numeral in base N notation to one in base N^2 notation and vice versa.

To change 22221_3 to base nine notation, begin at the right and group the digits by 2's; change each numeral represented by each group of digits to a base nine numeral.

$$\begin{tikzcd} 2_3 \arrow[d] & 22_3 \arrow[d] & 21_3 \arrow[d] \\ 2_9 & 8_9 & 7_9 \end{tikzcd} \quad \text{or } 22221_3 = 287_9$$

To change from base N^2 notation to base N notation, represent each digit in the base N^2 numeral in base N notation.

To change 354_9 to base three notation, represent 3_9, 5_9, 4_9, in base three notation.

$$\begin{tikzcd} 3_9 \arrow[d] & 5_9 \arrow[d] & 4_9 \arrow[d] \\ 10_3 & 12_3 & 11_3 \end{tikzcd} \quad \text{or } 354_9 = 101211_3$$

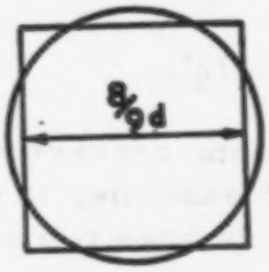
15 The Rhind Papyrus

One of the oldest mathematical documents, the Rhind Papyrus, was copied about 1650 B.C. by a scribe named Ahmes. The first part is a collection of recreational problems, and the second a record of the mathematics of that time. Notice the method of multiplication used by the Egyptians.

RHIND OR AHMES PAPYRUS. (C 1650 B.C.)

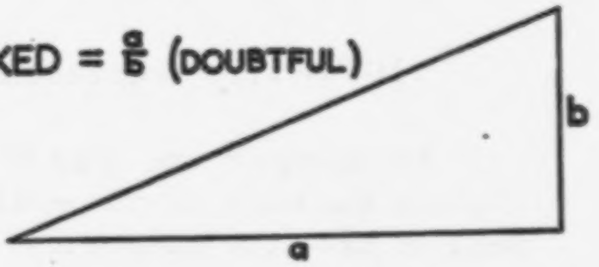
ADDITION

SUBTRACTION



AREA OF CIRCLE $= \frac{64}{81} d^2$

SEKED $= \frac{a}{b}$ (DOUBTFUL)



MULTIPLICATION BY DOUBLING.

115 x 23

- 1	115
- 2	230
- 4	460
8	920
- 16	1840

PRODUCT = 115+230+460+1840 = 2645

DIVISION BY DOUBLING.

364 ÷ 16

1	16
-2	32
-4	64
8	128
-16	256

364 = 256+64+32+12
= 16.16+4.16+2.16+12

364÷16
= 22 REMAINDER 12

The Rhind or Ahmes Papyrus

" Seven houses ; in each are seven cats ; each cat kills 7 mice ; each mouse would have eaten seven ears of spelt ; each ear of spelt will produce seven hekats of grain. What is the total of all these ? "

This problem, taken from the Rhind Papyrus, is the earliest version of the riddle " As I was going to St. Ives ".

If ever you go to the British Museum, be sure you see this papyrus. It was written by Ahmes about 1650 B.C., and was brought to England by an Egyptologist named Rhind in the middle of the nineteenth century. It is believed that Ahmes copied from an older work, now lost. The papyrus has the rather odd title " Directions for knowing all dark things ".

The papyrus can be divided into two main sections. There are problems of a recreational and utilitarian nature, of which the above is an example. But, much more important, it contains a record of the then known Mathematics.

Many of the formulae given are only approximate, and often refer to special cases, as, for example, in the case of the formulae for the areas of a triangle and a trapezium: $\Delta = \frac{1}{2}$ base × side, and trapezium $= \frac{1}{2}$ (sum of parallel sides) x one non-parallel side. It does give quite a good approximation for the area of a circle : " diminish the diameter of the circle by one-ninth of itself, and the square on the remainder is the area of the circle ". This gives $\pi = (2 \times \frac{8}{9})^2 = 3\frac{13}{81} = 3.1605$, which is only about 0.6% too high.

EXERCISES

1. Using base 5 notation, solve the following:

(a) 234
123
342

(b) 3412
- 2343

(c) 3402
x 43

(d) 43 $\overline{)322341}$

2. Given the addition and multiplication tables below:

+	0	1	∧	△	□
0	0	1	∧	△	□
1	1	∧	△	□	10
∧	∧	△	□	10	11
△	△	□	10	11	1∧
□	□	10	11	1∧	1△

x	0	1	∧	△	□
0	0	0	0	0	0
1	0	1	∧	△	□
∧	0	∧	□	11	1△
△	0	△	11	1□	∧∧
□	0	□	1△	∧∧	△1

Solve the following problems:

(a) □ ∧
+ △ △

(b) □ □ 1
- ∧ ∧ △

(c) △ □ ∧
x ∧

(d) $\frac{∧∧}{□}$

3. Using base 2 notation, solve the following:

(a) 1101
11
111

(b) 1011
- 101

(c) 1011
x 101

(d) 101 $\overline{)110111}$

4. Change each of the following to a numeral in the indicated base:

a. $11011_2 = \_\_\_\__{10}$

b. $2143_5 = \_\_\_\__{10}$

(c) $77_{10}\ \_\_\_\__6$

(d) $86_{10}\ \_\_\_\__3$

5. Using exponents, express each of the following in simpler form:

a. $3^4 \cdot 3^2$

b. $3^0 \cdot 4^0$

c. $6^8 \cdot 6^2 \div 6^5$.

d. $2^{-3} \cdot 2^5$

e. $\left(\frac{3}{4}\right)^0$

f. $(3862)^0$

EXERCISES (answers) **UNIT II**

1. a. 1304
   b. 1014
   c. 322341
   d. 3402

2. a. | △ ○

   b. ∧ | △

   c. | ∧ △ □

   d. △

3. a. 10111
   b. 110
   c. 110111
   d. 1011

4. a. 27
   b. 298
   c. 205
   d. 10012

5. a. 3^6

   b. 1

   c. 6^5

   d. 2^2

   e. 1

   f. 1

ASSIGNMENT

1. Express exponentially:

   a. $4 \times 4 \times 4 \times 4 \times 4$

   b. $2 \times 2 \times 2$

   c. $6 \times 10 \times 10 \times 10 \times 10 \times 10$

   d. $3 \times 4 \times 4 \times 3 \times 4 \times 4$

2. Solve the following using exponents:

   a. $2^6 \times 2^4$

   b. $X^4 \cdot X^2$

   c. $3^6 \div 3^2$

   * d. $2^4 \times 2^8 \div 2^{10}$

   * e. $4^2 \cdot 2^5$

   f. $8^5 \div 8^5 \times 8^5$

   g. $5^{-3} \div 5^{-3}$

   h. $6^0 \times 6^1$

   i. $(2/3)^0$

3. Find the errors in each of the following numerals:

   a. 464_5 b. 296_4 c. 3086_6

4. Write to 20 using the symbol □ for zero, ǀ for one, ∧ for two, and △ for three.

5. Given the addition and multiplication tables below:

+	□	ǀ	∧	△
□	□	ǀ	∧	△
ǀ	ǀ	∧	△	ǀ□
∧	∧	△	ǀ□	ǀǀ
△	△	ǀ□	ǀǀ	ǀ∧

x	□	ǀ	∧	△
□	□	□	□	□
ǀ	□	ǀ	∧	△
∧	□	∧	ǀ□	ǀ∧
△	□	△	ǀ∧	∧ǀ

Solve the following problems:

(a)
∧△ǀ
ǀ□∧
+ △∧□

(b)
△∧△
− ǀ△□

(c)
∧△ǀ
× ∧△

(d) ∧△ ⟌ ǀ△∧△△

ASSIGNMENT

6. Change each of the following to a numeral in the indicated base:

a. $37_{10} = \_\_\__{4}$

b. $423_5 = \_\_\__{10}$

c. $11011_2 = \_\_\__{10}$

d. $382_9 = \_\_\__{10}$

e. $964_{10} = \_\_\__{8}$

f. $52_6 = \_\_\__{10}$

g. $35_{10} = \_\_\__{6}$

h. $82_{10} = \_\_\__{5}$

i. $75_{10} = \_\_\__{2}$

*j. $110.11_2 = \_\_\__{10}$

7. Write to 20 in binary notation.

8. Solve the following problems in base 2:

a. $\begin{array}{r} 1011 \\ 111 \\ 1011 \\ 110 \\ \hline \end{array}$

b. $\begin{array}{r} 101 \\ 11 \\ 101 \\ \hline \end{array}$

c. $\begin{array}{r} 1101 \\ -\ 111 \\ \hline \end{array}$

d. $\begin{array}{r} 10100 \\ -\ 1011 \\ \hline \end{array}$

e. $\begin{array}{r} 1101 \\ \times\ 101 \\ \hline \end{array}$

f. $\begin{array}{r} 110 \\ \times\ 11 \\ \hline \end{array}$

g. $11\overline{)1011101}$

h. $101\overline{)1101110}$

9. In what base is each of the following a true sentence:

a. $3 \times 4 = 22$

b. $3 + 6 = 12$

c. $6 \times 8 = 40$

d. $2 + 3 + 2 = 12$

e. $12 - 4 = 5$

f. $10 \div 3 = 3$

10. Multiply 18 x 57 by the Russian peasant method. See p. 34.

ASSIGNMENT (answers)

1. a. 4^5

   b. 2^3

   c. $6 \cdot 10^5$

   d. $3^2 \cdot 4^4$

2. a. 2^{10}

   b. X^6

   c. 3^4

   d. 2^2

   e. $(2^2)^2 = 2^4 \cdot 2^5 = 2^9$

   f. 8^5

   g. 1

   h. 6

   i. 1

3. a. In base 5, there is no digit 6.

   b. In base 4, digits 9 and 6 not used

   c. In base 6, digits 6 and 8 not used.

4.
   1. |
   2. ∧
   3. △
   4. |□
   5. ||
   6. |∧
   7. |△
   8. ∧□
   9. ∧|
   10. ∧∧
   11. ∧△
   12. △□
   13. △|
   14. △∧
   15. △△
   16. |□□
   17. |□|
   18. |□∧
   19. |□△
   20. ||□

5. a. | △ | △

   b. | △ △

   c.

$$\begin{array}{ccccc} & & \wedge & \triangle & | \\ & & & \wedge & \triangle \\ \hline & \wedge & \square & | & \triangle \\ | & | & \wedge & \wedge & \\ \hline | & \triangle & \wedge & \triangle & \triangle \end{array}$$

   d.

$$\begin{array}{rccccc} & & & \wedge & \triangle & | \\ \wedge\,\triangle\,\big) & \overline{|} & \overline{\triangle} & \overline{\wedge} & \overline{\triangle} & \overline{\triangle} \\ & \underline{|} & \underline{|} & \underline{\wedge} & & \\ & & \wedge & \square & \triangle & \\ & & \underline{\wedge} & \underline{\square} & \underline{|} & \\ & & & & \wedge & \triangle \\ & & & & \underline{\wedge} & \underline{\triangle} \end{array}$$

ASSIGNMENT (answers) UNIT II

6. a. 211
   b. 113
   c. 27
   d. 317
   e. 1704
   f. 32
   g. 55
   h. 312
   i. 1001011
   j. 6.75

7. 1. 1
   2. 10
   3. 11
   4. 100
   5. 101
   6. 110
   7. 111
   8. 1000
   9. 1001
   10. 1010
   11. 1011
   12. 1100
   13. 1101
   14. 1110
   15. 1111
   16. 10000
   17. 10001
   18. 10010
   19. 10011
   20. 10100

8. a. 100011
   b. 1101
   c. 110
   d. 1001
   e. 1000001
   f. 10010
   g. 11111
   h. 10110

9. a. Base 5
   b. Base 7
   c. Base 12
   d. Base 5
   e. Base 7
   f. Base 9

10. 18	x	57	
* 9		114	
4		228	114
2		456	+912
* 1		912	1026

UNIT III

SETS AND SENTENCES

UNIT III. SETS AND SENTENCES

"The time has come," the Walrus said,
"To talk of many things:
Of shoes--and ships--and sealing-wax--
Of cabbages--and kings--
And why the sea is boiling hot--
And whether pigs have wings."

-----Lewis Carroll

SENTENCES AND STATEMENTS

A statement is the idea expressed by a complete sentence. The statement or idea represented by the sentence may be either true or false. Thus $6 + 4 = 9$ represents a false statement while $3 + 2 = 5$ represents a true statement.

OPEN SENTENCES

Sentences written as $3 + 2 = ?$ or as $3 + \square = 7$ are open sentences, since it is impossible to tell whether they express true or false ideas until they are completed. The "?" or "$\square$" or any symbol so used is a placeholder. In algebra letters are often used as placeholders in open sentences. A letter or other symbol used as a placeholder for an element or elements of a set is called a variable. An open sentence such as $3 + 2 = \square$ is traditionally called an equation, while open sentences such as $3 + 2 \neq \square$, $3 + 2 < \square$, and $3 + 2 > \square$ are traditionally called inequalities.

UNIVERSAL SET

For each open sentence there is a set of objects (or elements) each of which may be used instead of the placeholder to make the sentence express a true or a false statement. This set designated by U is called the universal set or the universe for the given situation.

SOLUTION SET

The set of objects (or elements) of the universe or universal set that will make a given open sentence a true statement is called the solution set or truth set.

Illustration: $U = \{1, 2, 3, 4, \cdots\}$
The solution set for $\square + 6 < 10$ is $\{1, 2, 3\}$
The solution set for $\square + 6 = 10$ is $\{4\}$

SUBSETS

A solution set or truth set is a subset of the given universal set. Set A is said to be a subset of set B if an only if each element of set A belongs to set B. The subset A may contain all of the elements or only some of the elements or none of the elements of the given set B. If a set contains no elements, it is called the empty or null set and written as $\emptyset$ or { }. By definition the null set is a subset of every set.

Set A is said to be a proper subset of set B if each element of set A is also an element of set B, and if set B has at least one element which is not in set A.

If U = {a, b, c} the subsets would include the following:

A = {a, b, c}
B = {a, b}
C = {a, c}
D = {b, c}
E = {a}
F = {b}
G = {c}
H = { } or $\emptyset$

In symbols D is a subset of U is written "$D \subset U$" and is read "D is a subset of U". "$U \supset D$" is read "U contains D". The proper subsets of U include all the above except {a, b, c}. It can be proved that for every set of n elements there are exactly 2^n subsets. If a set contains 3 elements, it has 2^3 or 8 subsets. If a set contains 4 elements it has 2^4 or 16 subsets.

Given the universal set U = {January, February, March, April, May, June} find the solution set for the open sentence: ▭ has 29 days in Leap Year. The solution set is {February}.

PROBLEMS

Find solution sets for the following and check your results with those on page 51.

1. If U equals {Mary, Jane, John, Phillip, James} find the solution set for each of the following open sentences:
   a. All boys who have names starting with 'J' include ▭.
   b. All boys who do not have names starting with 'J' are □.
   c. All persons who do not have names starting with 'J' include △.
2. Write all the proper subsets of {1, 2, 3, 4}.
3. Write all the subsets of {Bill, Harry, Mike, Jim}.

NOTATION FOR ELEMENT OF A SET

The symbol "$\in$" is read "is an element of." The symbol "$\notin$" is read "is not an element of". Thus "$4 \in A$" is read "4 is an element of A", and "$a \notin M$" is read "a is not an element of M".

If $A = \{1, 2, 3, 4\}$, then $4 \in A$ but $6 \notin A$.

EQUIVALENT AND EQUAL SETS

If two sets have the same cardinal number associated with them, their elements can be placed in a one-to-one correspondence. Such sets are called equivalent sets. $\{1, 2, 3\}$ and $\{a, b, c\}$ are equivalent sets since each set contains the same number of elements.

Equivalent sets that contain the same elements are called equal sets. For example, $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are equal sets. Equal sets are always equivalent sets, but equivalent sets are not necessarily equal sets.

DISJOINT SETS

Disjoint sets are sets having no elements in common. If set $A = \{5, 6, 7\}$ and set $B = \{8, 9, 10\}$, the sets are equivalent but disjoint. The set of all even numbers and the set of all odd numbers furnish another example of disjoint sets.

UNION OF SETS

Consider the sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{3, 6, 9\}$. Find a set C that contains all the elements in either A or B or in both A and B. Set $C = \{1, 3, 5, 6, 7, 9\}$ and the operation used to find C is called "the union of sets A and B". A union B equals C is written as "$A \cup B = C$".

If set $M = \{4, 6, 8, 10\}$ and set $N = \{3, 6, 9, 12\}$, then $M \cup N = \{3, 4, 6, 8, 9, 10, 12\}$.

INTERSECTION OF SETS

Again consider the set $A = \{1, 3, 5, 7, 9\}$ and the set $B = \{3, 6, 9\}$. Find a set D containing only those elements common to both A and B. Set D = {3, 9 and the operation used to find D is called intersection. "A intersection B = D" is written "$A \cap B = D$".

If set $M = \{4, 6, 8, 10\}$ and set $N = \{3, 6, 9, 12\}$, then $M \cap N = \{6\}$.

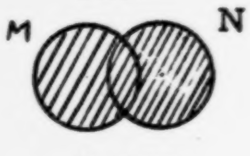
When two sets have no elements in common, their intersection is the empty or null set. If set $R = \{2, 3, 4\}$ and set $S = \{20, 30, 40, 50, 60\}$, then $R \cap S = \{\ \}$.

REPRESENTATION OF UNION AND INTERSECTION BY VENN DIAGRAMS

Venn diagrams, so called because they were popularized by the English mathematician and logician John Venn, are often used to represent the union or the intersection of sets.

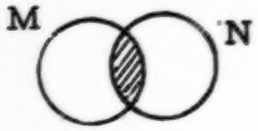
(A) If set M and set N have some elements in common, the circles shown below represent the union and the intersection of the sets. The shaded areas represent the solution sets.

Union



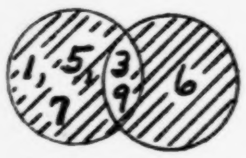
$M \cup N$

Intersection



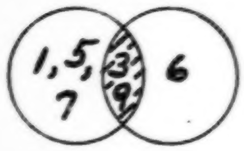
$M \cap N$

Illustrations of union and intersection using the sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{3, 6, 9\}$ follow:



$A \cup B = C$

$C = \{1, 3, 5, 6, 7, 9\}$

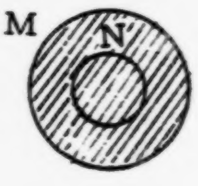


$A \cap B = C$

$C = \{3, 9\}$

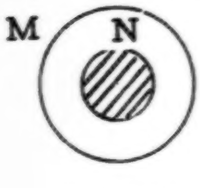
B) If set N is a proper subset of set M, the circles shown below represent the union and the intersection of the sets.

Union



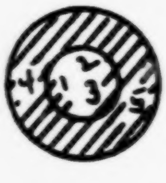
$M \cup N$

Intersection



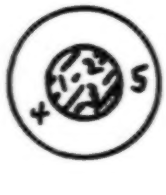
$M \cap N$

Illustrations of union and intersection using the sets $P = \{1, 2, 3, 4, 5\}$ and $Q = \{1, 2, 3\}$ follow:



$P \cup Q = \square$

$\square = \{1, 2, 3, 4, 5\}$



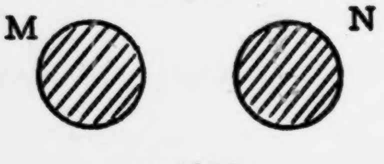
$P \cap Q = \triangle$

$\triangle = \{1, 2, 3\}$

2

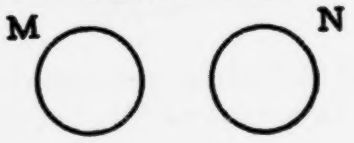
C) If set M and set N have no elements in common, the circles shown below represent the union and the intersection of the sets.

Union



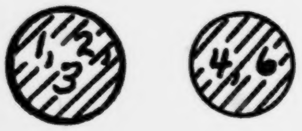
M ∪ N

Intersection

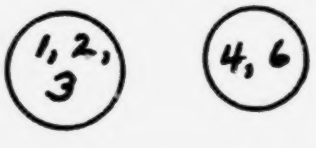


M ∩ N

Illustrations of union and intersection using the sets R = {1, 2, 3} and S = {4, 6} follow:



R ∪ S = T
T = {1, 2, 3, 4, 6}



R ∩ S = V
V = { } or Φ

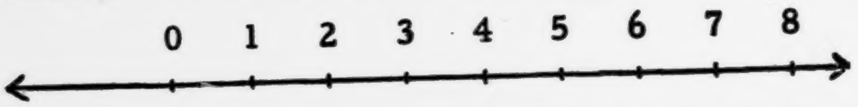
PROBLEMS

Solve these problems using Venn diagrams to show union and intersection.

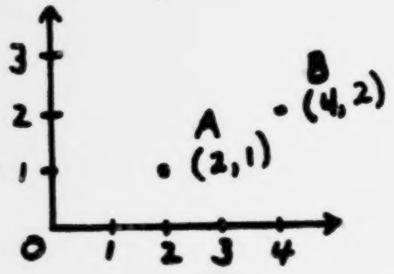
4. If A = {2, 4, 6, 8} and B = {1, 3, 5, 7}, find A ∪ B and A ∩ B.
5. If P = { ▲, □ } and Q = { △, ▼, ▯ } find P ∪ Q and P ∩ Q.
6. If A = {Jim} and B = {Helen, Ann}, find A ∪ B and A ∩ B.
7. If M = {1, 3, 4, 6} and N = {1, 3, 4}, find M ∪ N and M ∩ N.

U x U

It can be shown that a single number corresponds to each point on the number line and that each point on the number line corresponds to a single number. The point which corresponds to the number 2 is called the graph of 2 and the number 2 is called the coordinate of the point.

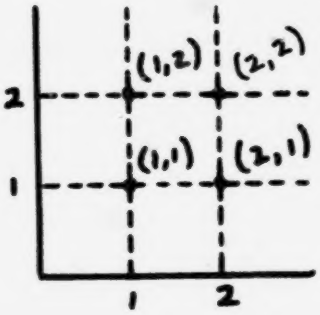


If a pair of number lines intersect at right angles as shown in the illustration, then a pair of numbers must be used to locate a point. To locate point A, begin at 0, go two spaces to the right on the horizontal number line, and then move up one space. The point A is designated by the ordered pair of numbers (2, 1) called the coordinates of point A. The point marked B has the coordinates (4, 2).



Consider a universe of 2 numbers, U = {1, 2}. The set of all ordered pairs that can be formed using these elements is {(1, 1), (1, 2), (2, 1), (2, 2)}. This set is denoted by "UxU" and is read "U cross U".

The diagram below represents the set of number pairs formed from U = {1, 2} and is referred to as the graph of U x U.



USE OF SETS IN ARITHMETIC

The language of sets is easy for students to learn and children enjoy using set notation. Sets may be used to help clarify some of the ideas in arithmetic. They may also be used for novel review.

Here are some typical examples from arithmetic:

1. Form the set of prime numbers less than 20.

   Solution: {2, 3, 5, 7, 11, 13, 17, 19}

2. Given the set of decimal fractions {.6, 2.4, .125},

   Rewrite this set using per cents.

   Solution: {60%, 240%, 12.5%}

3. What is the smallest number represented in set C if C = {2/3, 2/5, 2/4, 2/7, 2/6}?

   Solution: 2/7

4. Find the solution set for the open sentence 5 + ☐ < 8 if U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

   Solution set: {0, 1, 2}

5. Write a set consisting of 5 fraction numerals which represent the number 1/2. Is there only one such set?

   Solution: {'2/4', '4/8', '5/10', '6/12', '7/14'} No.

ANSWERS

1. a. {John, James}
   b. {Phillip}
   c. {Mary, Phillip}

2. {1, 2, 3}
{1, 2, 4}
{1, 3, 4}
{2, 3, 4}
{1, 2}
{1, 3}
{1, 4}
{2, 3}
{2, 4}
{3, 4}
{1}
{2}
{3}
{4}
{ }

3. {Bill, Harry, Mike, Jim}
{Bill, Harry, Mike}
{Bill, Harry, Jim}
{Bill, Mike, Jim}
{Harry, Mike, Jim}
{Bill, Harry}
{Bill, Mike}
{Bill, Jim}
{Harry, Mike}
{Harry, Jim}
{Mike, Jim}
{Bill}
{Harry}
{Mike}
{Jim}
{ }

4. a. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   b. $A \cap B = \{\ \}$

5. a. $P \cup Q$ = { ▲, □, ▼, ▯, △ }
   b. $P \cap Q = \{\ \}$

6. a. $A \cup B$ = {Jim, Helen, Ann}
   b. $A \cap B = \{\ \}$

7. a. $M \cup N = \{1, 3, 4, 6\}$
   b. $M \cap N = \{1, 3, 4\}$

SUMMARY OF SYMBOLISM

Symbol	Interpretation
$\{a, b\}$	the set whose elements are a and b
$\{\ \}$ or ϕ	the empty or null set
U	the universe or universal set
$U \times U$	U cross U
$a > b$	a is greater than b
$a \geqslant b$	a is greater than or equal to b
$a < b$	a is less than b
$a \leqslant b$	a is less than or equal to b
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$M \subset N$	set M is a subset of set N or set M is contained in set N
$M \supset N$	set M contains set N
$x \in M$	x is an element of set M
$x \notin M$	x is not an element of set M
$A \cup B$	A union B
$A \cap B$	A intersection B

REFERENCES FOR TEACHERS

Adler--The New Mathematics: pp. 35-40

Adler--Mathematics: Exploring the World of Numbers and Space: pp. 50-53

Banks--Elements of Mathematics: pp. 232-235

Johnson and Glenn--Sets, Sentences and Operations: pp. 2-57

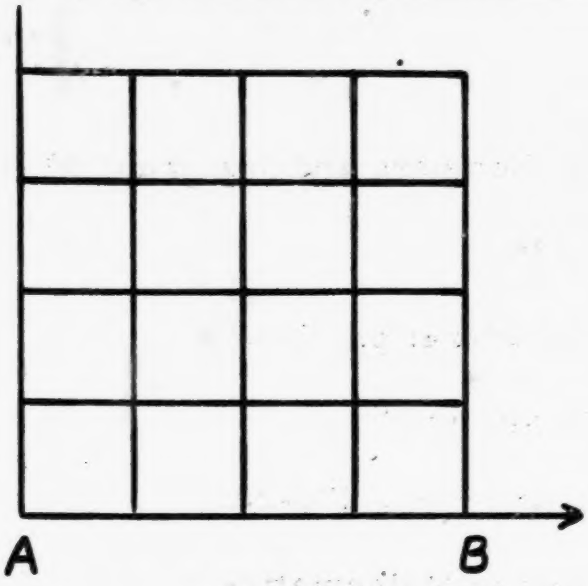
Johnson and Glenn--Adventures in Graphing: pp. 1-59

NCTM, 25th Yearbook--Instruction in Arithmetic: pp. 276-289

Osborn, DeVault, Boyd, and Houston--Extending Mathematics Understanding: pp. 97-114

Swain--Understanding Arithmetic: pp. 27-50

3.01 Tic - Tac - Toe (Locating Points in the Plane)



A game similar to tic-tac-toe can be used to teach primary and elementary children to locate points in the plane.

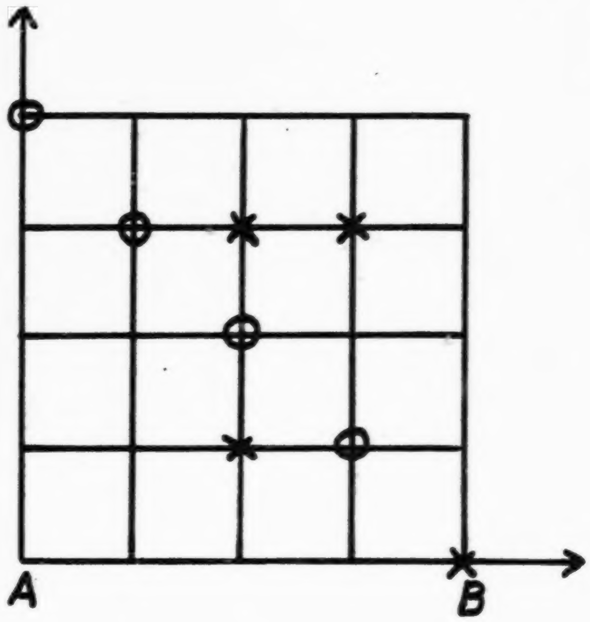
Draw a lattice on the board as in the illustration. To locate a point, count from A, the zero point, to the right and then up. A child names a pair of numbers to locate each point.

Divide the class into two teams, and let members of each team take turns naming a point. A point may be indicated by an "x" for one team and by an "o" for the other. If a child makes a mistake, he loses the turn for his team. The team that can get a specified number of points in a straight line wins. Variations of this game can include more points or a lattice that will include negative numbers.

Suppose that the team getting four points in a line wins. A sample game for a second grade may be as follows:

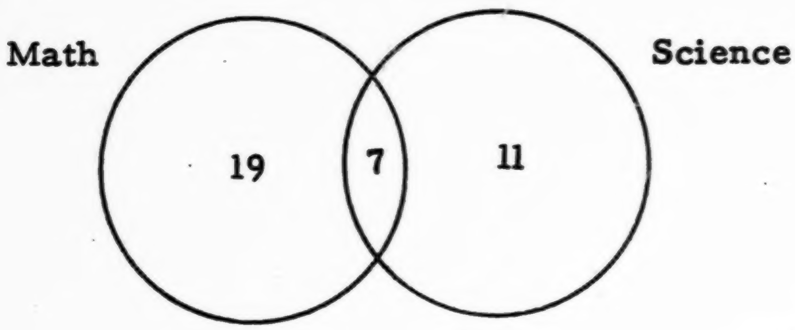
x team	o team
(2, 3)	(3, 1)
(2, 1)	(2, 2)
(3, 3)	(1, 3)
(4, 0)	(0, 4)

The o team is the winner. The arrows indicate the order in which the points were named.

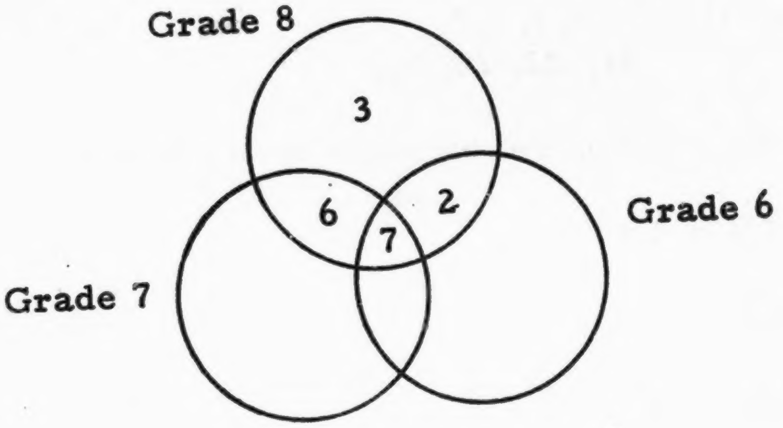


3.02 Problems Easily Solved with Venn Diagrams:

a. In a class of 37 students each of whom is enrolled in either mathematics or science, 26 are enrolled in mathematics and 7 in both mathematics and science. How many are in science? (18)



b. There are 18 patrol boys in the 8th grade in Luna School this year. Of these, 13 were patrol boys in Grade 7; 9 were patrol boys in Grade 6; and 7 were patrol boys in Grades 6 and 7. How many of the 18 boys have not previously served as patrol boys? (3)



3.03 Pictures from Ordered Pairs

Use the following ordered pairs to draw a map of Illinois.

(1,28)
(1,26)
(2,25)
(2,24)
(6,20)
(6,17)
(7,16)
(8,16)
(9,17)
(10,16)
(10,13)
(9,12)
(9,10)
(11,9)
(12,8)
(13,7)
(16,6)
(16,4)
(15,3)
(15,2)
(16,0)
(17,1)
(18,2)
(19,3)
(21,2)
(22,1)
(23,2)
(23,4)
(24,5)
(25,6)
(24,7)
(25,8)
(25,11)
(26,13)
(27,14)
(28,15)
(28,17)
(27,18)
(27,20)
(28,21)
(28,41)
(27,42)
(26,44)
(25,46)
(25,48)
(6,48)
(9,46)
(10,44)
(10,43)
(9,42)
(8,41)
(4,39)
(5,36)
(5,35)
(4,34)
(4,33)
(2,31)
(2,29)
(1,28)

Use graph paper to let students create other pictures that may be reproduced by using coordinates.

04 Frame Problems

Problems such as the following are included in modern elementary programs. These can be adapted for use in the primary grades.

Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$

Find the solution set for each open sentence:

1 $\square + 2 = 7$

2 $\square - 3 = 4$

3 $\triangle + \triangle - 2 = 6$

4 $\square + 3 \times \square = 8$

5 $\square + 2 < 15$

6 $\triangle - 2 > 15$

7 $3 \times \square < 21$

8 $2 \times \square + 1 > 4$

9 $5 < \triangle < 16$

10 $1 < \square + 2 < 12$

For more gifted groups the following types may be used:

11. $(\square \times \square) - (7 \times \square) + 12 = 0$ $\{3, 4\}$

12. $(\triangle \times \triangle) - (8 \times \triangle) + 12 = 0$ $\{2, 6\}$

13. $(\square \times \square) - (6 \times \square) + 5 = 0$ $\{1, 5\}$

14. $(\square \times \square) - (5 \times \square) + 6 = 0$ $\{2, 3\}$

What is the sum of the numbers in the solution set? What is the product of these numbers? Do you find the sum and product represented in the open sentence? Can you make use of these facts to shorten your work in finding the solution set for:

$$(\square \times \square) - (7 \times \square) + 10 = 0$$

3.05 Open Sentences for Drill

Interesting drill on the addition combinations can be provided through open sentences such as the ones suggested in this exercise. Each solution will be a pair of numbers. Using the set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ find all possible solutions for:

$\square + \triangle = 12$ $\square - \triangle = 6$

3.06 Identifying Sets

Use file cards or cut 2" x 3" or 3" x 5" cardboard cards and draw or represent elements of a set on one side. On similar cards of a different color record a description of the set. A game may be played with children pairing the proper cards.

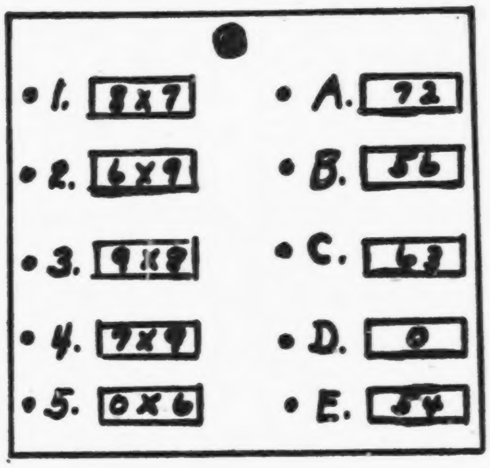
Typical sets and descriptions might include:

{2, 4, 6, 8... }	The set of even numbers
{1, 3, 5, 7... }	The set of odd numbers
{2, 3, 5, 7, 11... }	The set of prime numbers
{0, 5, 10, 15, 20... }	The set of multiples of 5
{0, 6, 12, 18, 24... }	The set of multiples of 6

The game can include naming the next number in the set.

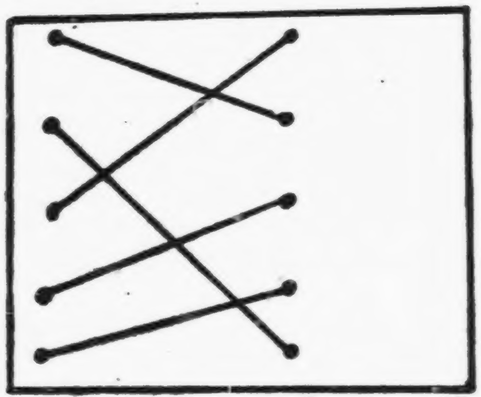
Variations may include sets of animals, dishes, books, etc. for the younger children; and sets such as the set of integers, the set of rational numbers, the null set, etc. for older students.

3.07 Math Quizzer



Use a piece of plywood 20" or 30" x 16".

Install terminals to the left of each card in the position shown by the dots in Fig. I. Use metal corners or brads to hold the card. Prepare several sets of cards. They may be changed daily. The cards are placed in such a way that making contact with the correct terminal completes the circuit and lights the bulb. One wiring scheme is given below at the right, but the wiring pattern must be changed before students discover the pattern.



This device can be used by the slower student to learn basic facts such as 9 x 6 and 8 x 7 which he may find confusing.

3.08 Sets and Subsets

The set of numbers {9, 11, 16, 22, 8, 2, 18, 6, 3, 13, 15, 21} has some interesting subsets. If the numerals for these numbers are painted on blocks or squares of cardboard, identifying the various subsets becomes a game.

$\boxed{11}$ $\boxed{16}$ $\boxed{22}$ $\boxed{8}$ $\boxed{2}$

$\boxed{9}$ $\boxed{18}$

$\boxed{6}$ $\boxed{3}$ $\boxed{13}$ $\boxed{15}$ $\boxed{21}$

Making use of all twelve numbers, form six subsets each containing two numbers so that all have equal sums. Form four such subsets. Form three such subsets. Form two such subsets.

{9, 15} {20, 4} {6, 18} {16, 8} {21, 3} {22, 2}

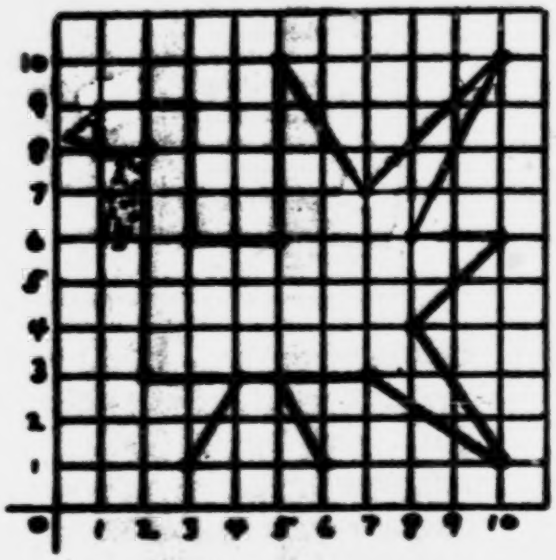
{22, 8, 6} {9, 11, 16} {18, 15, 3} {21, 2, 13}

{16, 22, 8, 2} {13, 15, 11, 9} {21, 18, 6, 3}

{16, 11, 18, 3, 15, 9} {6, 21, 8, 13, 22, 2}

3.09 Drawing Pictures Using Two Number Lines

On squared paper draw two number lines as shown in the diagram. Each corner of a square will represent a point that can be located by counting from 0 to the right along the horizontal number line and then counting up. For example, to locate the point (2, 3), begin at 0, count two spaces to the right along the horizontal number line, and then count up three spaces. Make a dot to represent the point.



The following set of points make a picture. Locate each point and represent it by a dot. Connect the dots to draw a picture.

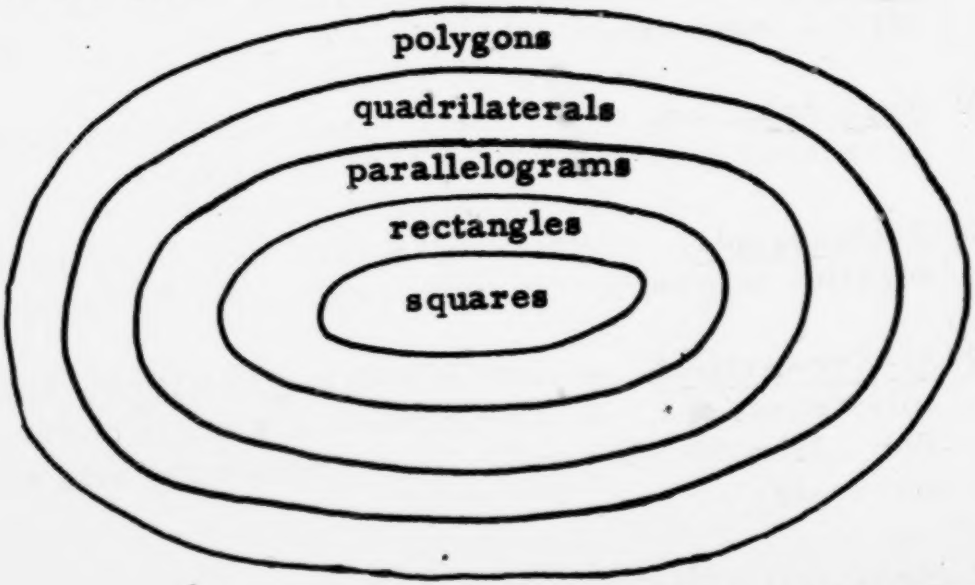
(2, 3), (7, 3), (10, 1), (8, 4), (10, 6), (8, 6), (10, 10), (7, 7), (5, 10), (5, 6), (3, 6), (3, 9), (1, 9), (1, 8), (2, 8), (2, 3), Draw line segments from (3, 1) to (4, 3) and from (5, 3) to (6, 1).

ACTIVITIES

3.10 Venn Diagrams

Use Venn diagrams to show that:

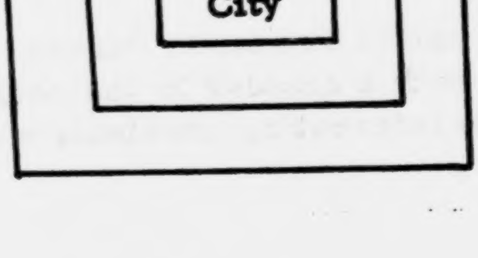
(a) the set of all squares is a subset of the set of all rectangles.
(b) the set of all rectangles is a subset of the set of all parallelograms.
(c) the set of all parallelograms is a subset of the set of all quadrilaterals.
(d) the set of all quadrilaterals is a subset of the set of all polygons.



Use the Venn diagram to write proper definitions of the geometic figures:

A square is a rectangle with two adjacent sides equal.
A rectangle is a parallelogram with one right angle.
A parallelogram is a quadrilateral with opposite sides parallel.
A quadrilateral is a polygon with four sides.
A polygon is a closed plane figure with straight sides.

Venn diagrams may be used in subjects other than in mathematics to show relationships.



ACTIVITIES

.11 Famous Mathematicians

c. 150 Ptolemy, an astronomer as well as a mathematician, wrote the famed "Almagest".

275 Diophantus, one of the first writers to use mathematical symbols, is noted for his work on equations. Diophantine equations are named in his honor.

410 Hypatia, first woman mathematician whose name is recorded, was supposedly stoned to death by an angry mob.

510 Aryabhata, one of the great Hindu mathematicians, was interested in arithmetic and progressions.

630 Brahmagupta, a Hindu mathematician, computed with negative numbers. He gave a definite rule for the quadratic.

820 Al-Khowarizmi, an Arab scholar, used Hindu numerals to solve equations. A corruption of his name became "algorism". For a long time algorism meant reckoning with Hindu-Arabic numerals.

850 Mahavira, a Hindu mathematician, stated that a number times 0 is 0. A noteworthy feature in his treatment of division of fractions was his rule to invert the divisor and multiply.

980 Gerbert, one of the first Christians to study in Moorish universities, later became Pope Sylvester II. Because of his study in Spain he introduced Hindu-Arabic numerals into Europe. However, he did not use the symbol for zero.

1100 Omar Khayyam, a great Persian poet and author of the Rubaiyat, wrote what was considered the best algebra produced by the Persians.

1150 Bhaskara, a Hindu mathematician, was interested in negative numbers. To console his daughter for an accident involving a pearl which had kept her from marrying, he wrote the Lilivati, a treatise on arithmetic and measurement. For his treatise on general arithmetic he had a name meaning "calculation of seeds".

1200 Fibonacci, also called Leonardo Pisano, engaged in a mathematical contest with a scholar in the court of Emperor Frederick in Sicily. Three interesting problems were given the contestants to solve.

EXERCISES

1. **Describe the following sets in words:**

   a. $A = \{2, 4, 6, 8, 10\}$
   b. $T = \{\text{January, June, July}\}$
   c. $M = \{1, 3, 5, 7, 9\}$
   d. $K = \{a, b, c, d\}$

2. **Are these sets equivalent?**

   a. $\{1, 3, 5, 7\}$ and $\{1, 7, 5, 3\}$
   b. $\{A, B, C, D\}$ and $\{1, 2, 3, 8\}$
   c. $\{2, 4, 6, 8\}$ and the set of even numbers less than 12

3. **If U. $= \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, write the following subsets:**

   a. The set of prime numbers
   b. The set of odd numbers
   c. The set of even numbers
   d. The set of natural numbers
   e. The set of perfect squares
   f. The set of perfect cubes
   g. The set of perfect fourth powers.

4. **If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, which of the statements expressed by the sentences below are true?**

   a. $2 \in A$
   b. $9 \notin A$
   c. $8 \in A$
   d. $2\ 1/2 \in A$

5. **Using Venn diagrams solve the following:**

   If $A = \{2, 4, 6, 8\}$ and $B = \{1, 4, 6, 7\}$ find

   a. $A \cup B$
   b. $A \cap B$

6. **Picture $(A \cap B) \cup C$ if A, B, and C are all subsets of U and all have some elements in common.**

7. **Show that $(A \cap B) \cap C = A \cap (B \cap C)$**

8. **Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$**
   Find solution sets for each of the following:

   a) $\square + 3 = 6$
   b) $\square + 12 < 18$
   c) $3 \times \square < 30$
   d) $\square - 6 < 10$
   e) $2 \times \square + 6 = 15$

EXERCISES (answers)

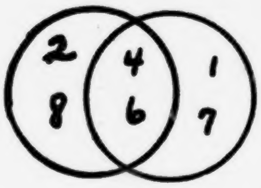
1. a. the set of even numbers less than 12
   b. the set of months whose names start with the letter J
   c. the set of odd numbers less than 11
   d. the first 4 letters of the alphabet

2. a. yes
   b. yes
   c. no

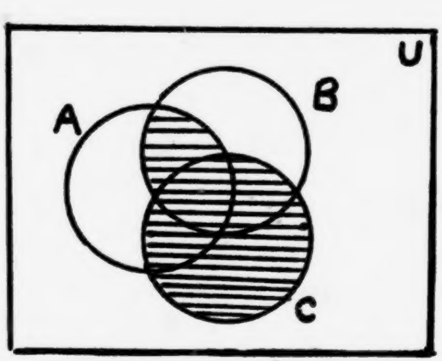
3. a. {2, 3, 5, 7, 11, 13}
   b. {3, 5, 7, 9, 11, 13}
   c. {2, 4, 6, 8, 10, 12}
   d. {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}
   e. {4, 9}
   f. {8}
   g. { } or ϕ

4. a. true
   b. true
   c. true
   d. false

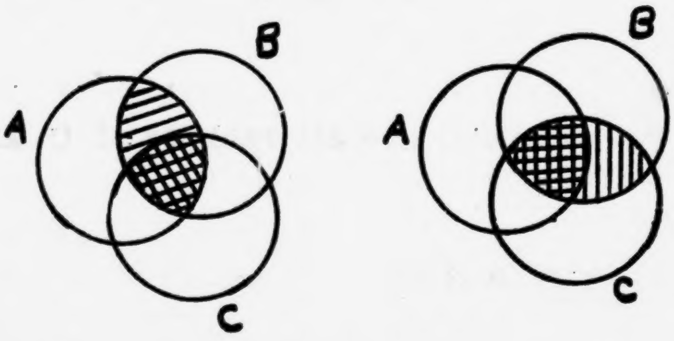
5. a. {1, 2, 4, 6, 7, 8}
   b. {4, 6}



6.



7.



8. a. {3}
   b. {1, 2, 3, 4, 5}
   c. {1, 2, 3, 4, 5, 6, 7, 8, 9}
   d. {7, 8, 9, 10, 11, 12, 13, 14, 15}
   e. { }

ASSIGNMENT UNIT III

1. Describe the following sets in words:

   a. $\{1, 3, 5, 7, 9, 11\}$
   b. {March, May}
   c. {Sunday, Saturday}
   d. $\{6, 12, 18, 24, 30, 36, \dots\}$

2. If $A = \{2, 3\}$ and $B = \{2, 6\}$, complete the following sentences to make a true statement.

   a. $\square \in A$ but $\notin B$
   b. $\square \in A$ and also $\in B$
   c. $\square \in B$ but $\notin A$

3. If $U = \{1, 4, 5, 6, 8, 9, 11, 15, 17\}$, form the following subsets:

   a. the set of odd integers
   b. the set of even integers
   c. the set of fractions
   d. the set of perfect squares
   e. the set of prime numbers

4. Given U = {'3/3', '3/4', '4/4', '4/6', '6/6', '6/8', '8/8'}

   a. Form a subset of values equivalent to 1.
   b. Form a subset of values equivalent to 3/4.
   c. Form a subset of fraction numerals in lowest terms.

5. Given the set $A = \{1, 4, 6, 8\}$ and $B = \{2, 4, 5, 7\}$ and $C = \{1, 3, 6, 7\}$

   a. Find $A \cup B$
   b. Find $A \cap B$
   c. Find $A \cap C$
   d. Find $B \cap C$
   e. Find $(A \cap B) \cap C$
   f. Find $A \cap (B \cap C)$

6. Let $U = \{0, 1, 2, 3, 4, \dots\}$, $P = \{0, 2, 4, 6, 8, \dots\}$, $Q = \{1, 3, 5, 7, \dots\}$, and $R = \{5, 10, 15, 20, \dots\}$

   a. Find $P \cup Q$
   b. Find $P \cup R$
   c. Find $Q \cup R$
   d. Find $P \cap Q$
   e. Find $P \cap R$
   f. Find $Q \cap R$

ASSIGNMENT

7. A survey of the foreign language classes at Fullerton High School showed the following:

39 students were taking French
31 students were taking German
28 students were taking Spanish
8 students were taking French and German
12 students were taking German and Spanish
14 students were taking French and Spanish
5 students were taking all three languages

a. Draw three overlapping circles to describe this.

b. How many different students are taking a foreign language?

8. Given $U = \{1, 2, 3, \cdots\}$
Find the solution sets for each of the following:

a) $\square + \triangle = 10$ and $\square - \triangle = 4$

b) $\square \times \square - 12 \times \square + 27 = 0$

c) $2 \times \square + 3 \times \square + 5 = 20$

d) $\square + \square - 10 = 12$

e) $\square + 3 > 6$

f) $1 < \square + 5 < 30$

9. Given: Set P = {Ed, Tom, Jim, Joe}
Set Q = {Ed, Joe, Sam, John}
Set R = {Ed, John, Jim, Mike}

Answer the following questions and use Venn diagrams to check:

a) Is $P \cap (Q \cup R)$ equivalent to $(P \cap Q) \cup (P \cap R)$?

b) Is $P \cup (Q \cap R)$ equivalent to $(P \cup Q) \cap (P \cup R)$?

c) Is $(P \cap Q) \cap R$ equivalent to $P \cap (Q \cap R)$?

10. If $U = \{1, 2, 3\}$, construct and graph the set of all possible pairs of $U \times U$.

ASSIGNMENT (answers) UNIT III

1. a. the set of odd natural numbers less than 13
   b. the set of months whose names begin with M
   c. the set of all the days of the week whose names start with S
   d. the set of all positive numbers divisible by 6

2. a. 3
   b. 2
   c. 6

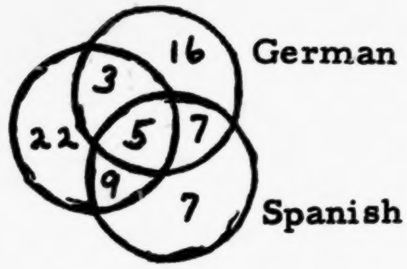
3. a. {1, 5, 9, 11, 15, 17}
   b. {4, 6, 8}
   c. { } or ∅
   d. {1, 4, 9}
   e. {5, 11, 17}

4. a. {'3/3', '4/4', '6/6', '8/8'}
   b. {'3/4', '6/8'}
   c. {'3/4'}

5. a. {1, 2, 4, 5, 6, 7, 8}
   b. {4}
   c. {1, 6}
   d. {7}
   e. { } or ∅
   f. { } or ∅

6. a. {0, 1, 2, 3, 4, ···}
   b. {0, 2, 4, 5, 6, 8, 10, 12, 14, 15, ···}
   c. {1, 3, 5, 7, 9, 10, 11, 13, 15, ···}
   d. { } or
   e. {10, 20, 30, ···}
   f. {5, 15, 25, ···}

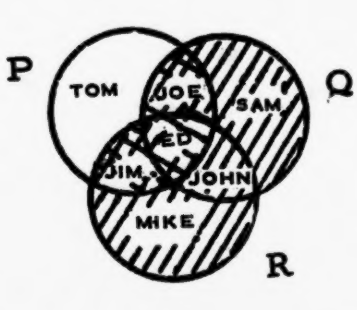
7. a.


French
German
Spanish
16
3
22
5
7
9
7

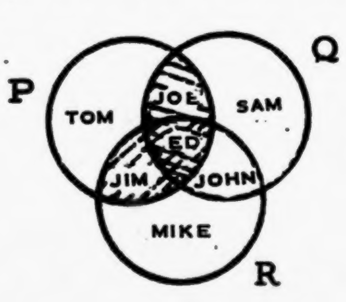
   b. 69

8. a) □ = 7 and △ = 3
   b) {9, 3}
   c) {3}
   d) {11}
   e) {4, 5, 6, ···}
   f) {1, 2, 3, ···, 22, 23, 24}

9. a. $P \cap (Q \cup R)$


P
Q
R
TOM
JOE
SAM
ED
JIM
JOHN
MIKE

$(P \cap Q) \cup (P \cap R)$

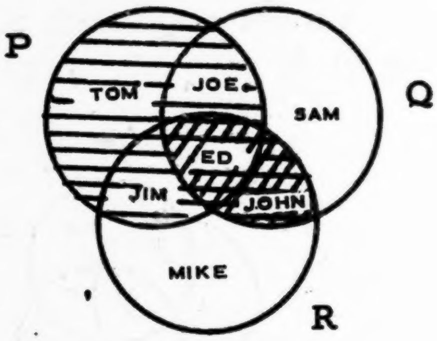

P
Q
R
TOM
JOE
SAM
ED
JIM
JOHN
MIKE

$P = \{Ed, Tom, Jim, Joe\}$
$Q \cup R = \{Ed, Joe, Sam, John, Jim, Mike\}$
$P \cap (Q \cup R) = \{Ed, Jim, Joe\}$

$P \cap Q = \{Ed, Joe\}$
$P \cap R = \{Jim, Ed\}$
$(P \cap Q) \cup (P \cap R) = \{Ed, Jim, Joe\}$

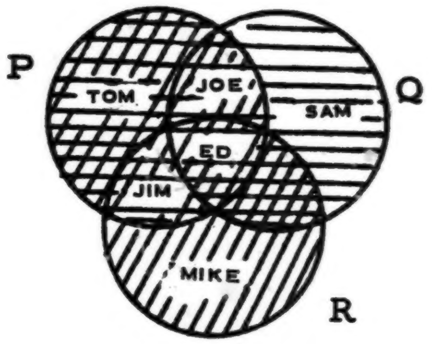
$P \cap (Q \cup R)$ is the same as $(P \cap Q) \cup (P \cap R)$

b) $P \cup (Q \cap R)$


P
Q
R
TOM
JOE
SAM
ED
JIM
JOHN
MIKE

$P = \{Ed, Tom, Jim, Joe\}$
$Q \cap R = \{Ed, John\}$
$P \cup (Q \cap R) = \{Ed, Tom, Jim, Joe, John\}$

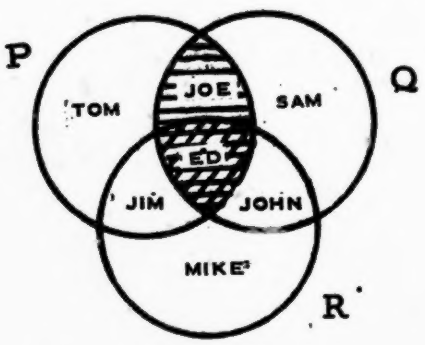
$(P \cup Q) \cap (P \cup R)$


P
Q
R
TOM
JOE
SAM
ED
JIM
MIKE

$(P \cup Q) = \{Ed, Tom, Jim, Joe, Sam, John\}$
$(P \cup R) = \{Ed, Tom, Jim, Joe, Mike, John\}$
$(P \cup Q) \cap (P \cup R) = \{Ed, Tom, Jim, Joe, John\}$

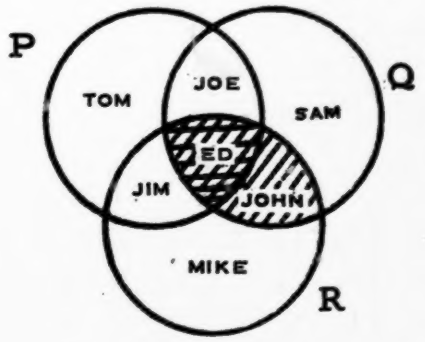
$P \cup (Q \cap R)$ is the same as $(P \cup Q) \cap (P \cup R)$

c. $(P \cap Q) \cap R$


P
Q
R
TOM
JOE
SAM
ED
JIM
JOHN
MIKE

$(P \cap Q) = \{Joe, Ed\}$
$R = \{Ed, John, Jim, Mike\}$
$(P \cap Q) \cap R = \{Ed\}$

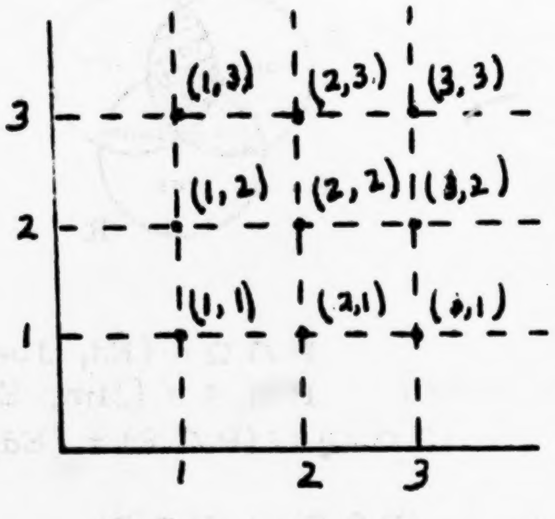
$P \cap (Q \cap R)$


P
Q
R
TOM
JOE
SAM
ED
JIM
JOHN
MIKE

$P = \{Ed, Tom, Jim, Joe\}$
$Q \cap R = \{Ed, John\}$
$P \cap (Q \cap R) = \{Ed\}$

$(P \cap Q) \cap R$ is the same as $P \cap (Q \cap R)$

10.


(1,3)
(2,3)
(3,3)
(1,2)
(2,2)
(3,2)
(1,1)
(2,1)
(3,1)
3
2
1
1
2
3

UNIT IV

OPERATIONS OF ADDITION AND MULTIPLICATION

UNIT IV. OPERATIONS OF ADDITION AND MULTIPLICATION

> *"What's one and one and one and one and one and*
> *one and one and one and one?"*
> *"I don't know," said Alice. "I lost count."*
> *"She can't do addition," said the Red Queen*
>
> *-----Lewis Carroll*

INTRODUCTION

In this unit some of the principles of the operations of addition and multiplication of natural numbers will be identified. Since modern courses in mathematics emphasize these basic principles as unifying concepts, this unit is of particular importance.

CONSTRUCTION OF THE SET OF ALL NATURAL NUMBERS

With the undefined elements 0 and 1 and the undefined operation +, the set of elements $\{0, 1, 1+1, 1+1+1, 1+1+1+1, \cdots\}$ is generated. Standard names have been assigned to these elements--zero, one, two, three, four, etc. This set is called the set of all natural numbers and designated by $N = \{0, 1, 2, 3, 4, \cdots\}$.

The operations of addition and multiplication in this unit will be applied to elements of the set of natural numbers.

OPERATION OF ADDITION

The word "addition", derived from the Latin "ad" meaning "to" and the Latin "dare" meaning "put", literally means "to put together".

Addition is closely related to the operation of union of sets. Recall that each set has a cardinal number associated with it. If two sets are disjoint (no elements in common), then addition is finding the cardinal number associated with the third set formed by the union of the two sets.

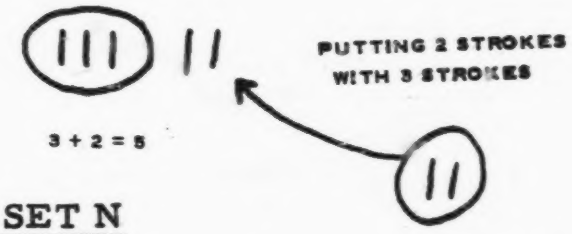
Consider the disjoint sets A = {Harry, Jim, Bill} and B = {Linda, Sue}. Then $A \cup B$ = {Harry, Jim, Bill, Linda, Sue}. Set A has the cardinal number 3 associated with it and may be represented by $n(A)=3$. Since $n(A)=3$ and $n(B)=2$, then $n(A \cup B) = n(A) + n(B)$ or $3 + 2 = 5$.

BINARY OPERATION

"Binary" means "two". Addition is called a binary operation because it is fundamentally an operation performed on two numbers called addends. In the language of mathematics $3 + 2$ indicates a binary operation of addition to be performed in a definite order on the two natural numbers 3 and 2. $3 + 2$ indicates that the operation is to be carried out by adding 2 to 3 resulting in the sum 5. This order of performing the operation is indicated in the vertical column:

$$\begin{array}{r} 3 \\ \underline{+\ 2} \end{array}$$

Ancient man probably indicated this by putting 2 strokes with 3 strokes he had previously made.



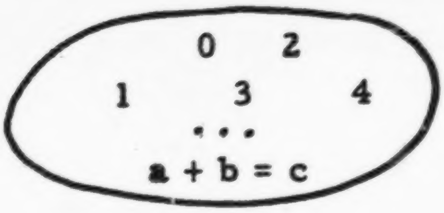
FUNDAMENTAL PRINCIPLES FOR THE SET N

In Unit I the four components of a mathematical system were listed: undefined terms, defined terms, assumptions, and theorems. The set of natural numbers was constructed using the undefined terms zero and one and the undefined operation addition.

In this unit the assumptions for the operations of addition and multiplication will include closure, commutativity, associativity, existence of identity elements, and distributivity of multiplication over addition. These assumptions are variously referred to by different writers as assumptions, postulates, axioms, principles, laws, or properties. In these units they will be called principles.

CLOSURE AND UNIQUENESS

It will be assumed that the sum of a pair of numbers in the set $N = \{0, 1, 2, 3, \cdots\}$ is a number in the given set N and that this sum is unique. For example, $8 + 3 = 11$, and 11 is also in N. Furthermore, since $8 + 3 = 11$ and only 11, the sum is unique.



When the binary operation of addition is performed on elements of the set $N = \{0, 1, 2, 3, \cdots\}$, the sums are unique numbers of the same set. For this reason the set N is said to be closed under the operation of addition.

The principles of closure and uniqueness for the set N under the operation of addition may be stated:

$$a + b = c$$

where a, b, and c are numbers in N and where c is uniquely determined by a and b.

COMMUTATIVE PRINCIPLE OF ADDITION

The order of addition of two numbers in the set N does not affect the sum. For example, $3 + 2 = 2 + 3$ since $3 + 2 = 5$ and $2 + 3 = 5$. This illustrates that addition is commutative.

The commutative principle of addition may be stated:

$$a + b = b + a$$

for every pair of numbers a and b in set N.

ASSOCIATIVE PRINCIPLE OF ADDITION

To add three or more numbers, the idea of a binary operation is merely extended. Thus $a + b + c$ must be defined as $(a + b) + c$ or $a + (b + c)$ with the understanding that the operations in parentheses are to be performed first. To add $2 + 3 + 4$, group the addends as $(2 + 3) + 4$ or as $2 + (3 + 4)$ and find the sum by performing 2 binary operations. To find the sum of $(2 + 3) + 4$, first add 3 to 2 and then add 4 to the sum 5. To find the sum of $2 + (3 + 4)$, first add 4 to 3 and then add the sum 7 to 2. Note that $5 + 4 = 2 + 7$.

This shows that the addends may be grouped in various ways and illustrates the associative principle of addition. Why can an addition problem be checked by adding in the opposite direction?

The associative principle of addition may be stated:

$$(a + b) + c = a + (b + c)$$

where a, b, and c are numbers in the set N.

EXISTENCE OF AN IDENTITY ELEMENT FOR ADDITION

The use of the digit 0 as a holder of place value in the decimal system of numeration as well as in systems of numeration with other bases has been discussed and some attention has been given to zero as a number.

Since $a + 0 = a$ for all a, zero is called the identity element for addition or the additive identity. For consistency $0 + 0$ is defined as 0.

The existence of an identity element for addition in the set N may be assumed:

$$a + 0 = 0 + a = a$$

where 0 and a are numbers in N.

This completes the principles to be introduced for the operation of addition on the numbers in set N. These principles will be applied to the addition of larger numbers.

ADDITION OF LARGER NUMBERS

It is possible to introduce a child in the primary grades to meaningful addition of larger numbers by demonstrating with real objects the principles of grouping and associativity. Sticks, tongue depressors, pennies, etc., may be grouped by ones, tens, and hundreds. In Unit II grouping was emphasized in the addition of numbers expressed in bases other than ten.

The ideas of grouping and associativity are basic to addition as is illustrated.

a) $12 + 6 = \square$
$(10 + 2) + 6 = \square$
$10 + (2 + 6) = \square$
$10 + 8 = 18$

b) $18 + 7 = \square$
$18 + (2 + 5) = \square$
$(18 + 2) + 5 = \square$
$20 + 5 = 25$

c. $54 + 32 + 46 = \square$
$(50 + 4) + (30 + 2) + (40 + 6) = \square$
$(50 + 30 + 40) + (4 + 2 + 6) = \square$
$120 + 12 = \square$
$120 + (10 + 2) = \square$
$(120 + 10) + 2 = \square$
$130 + 2 = 132$

The regrouping for example c by the usual method is illustrated below:

$$\begin{array}{r} 54 \\ 32 \\ 46 \\ \hline 132 \end{array} \qquad \begin{array}{c} \text{5 tens and 4 ones} \\ \text{3 tens and 2 ones} \\ \text{4 tens and 6 ones} \\ \hline \text{12 tens and 12 ones} \end{array} \qquad \begin{array}{c} 5\cdot10 + 4\cdot1 \\ 3\cdot10 + 2\cdot1 \\ 4\cdot10 + 6\cdot1 \\ \hline 12\cdot10 + 12\cdot1 \end{array}$$

The 12 ones are regrouped into one group of one ten and one group of 2 ones, and the 1 ten is grouped with the 12 tens. The 13 tens are regrouped into groups of 1 hundred and 3 tens. The sum represents groups of 1 hundred, 3 tens, and 2 ones.

The process of regrouping the 12 tens and 12 ones may also be illustrated as follows:

$$\begin{aligned} 12\cdot10 + 12\cdot1 &= 12\cdot10 + (1\cdot10 + 2\cdot1) \\ &= (12\cdot10 + 1\cdot10) + 2\cdot1 \\ &= 13\cdot10 + 2\cdot1 \\ &= (10\cdot10 + 3\cdot10) + 2\cdot1 \\ &= 1\cdot100 + 3\cdot10 + 2\cdot1 \\ &= 100 + 30 + 2 \\ &= 132 \end{aligned}$$

The "partial sums" method of addition, popular in the sixteenth century and sometimes referred to as the "lazy man's way of adding", also shows this regrouping.

$$\text{a)}\ \begin{array}{r} 12 \\ +\ 6 \\ \hline 8 \\ 10 \\ \hline 18 \end{array} \qquad \text{b)}\ \begin{array}{r} 18 \\ +\ 7 \\ \hline 15 \\ 10 \\ \hline 25 \end{array} \qquad \text{c)}\ \begin{array}{r} 54 \\ 32 \\ +\ 46 \\ \hline 12 \\ 120 \\ \hline 132 \end{array}$$

OPERATION OF MULTIPLICATION

The undefined operations of addition and multiplication on elements of the set of natural numbers may be compared. The addition 4 + 4 + 4 may be represented as 3 x 4. Multiplication can be symbolized in various ways:

1) 3 groups of 4

2) 3 four's

3) 3 times 4

4) 3 x 4

5) $\begin{array}{r} 4 \\ \underline{\times\ 3} \end{array}$

6) 4 multiplied by 3

7) multiply 4 by 3

8) 3·4

9) (3)(4)

Arrays, which are orderly arrangements of objects, may be used to help children discover the commutative principle of multiplication. A 3 x 4 array will have the same number of elements as a 4 x 3 array, but will present a different appearance.

```
      4                       3
    X X X X                 X X X
3   X X X X             4   X X X
    X X X X                 X X X
                            X X X
```

A 3 x 4 array has 3 rows of 4 objects and represents 4 + 4 + 4 while a 4 x 3 array has 4 rows of 3 objects and represents 3 + 3 + 3 + 3.

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

A relationship between multiplication and addition of natural numbers can be discovered by examining a multiplication table. In any column the number at the top is successively added to obtain the next number represented in the column. Counting by 2's, 5's, 6's, etc., also emphasizes the additive process.

FACTORS AND PRIME FACTORS

When two numbers are multiplied together, they are called factors or divisors of the product; e.g. if 4 x 3 equals 12, then 4 and 3 are spoken of as factors of 12. However, they are not prime factors, since 4 is not a prime number. The prime factors of 4 are 2 and 2. The prime factors of 12 are 2, 2, and 3.

Prime is derived from the Latin "primus" and means "first" or "fundamental". A prime number is any whole number greater than one that is exactly divisible only by itself and one. The first six prime numbers are 2, 3, 5, 7, 11, and 13. One is not considered a prime number. It is used as an undefined term.

A natural number that is the product of several primes is called a composite number.

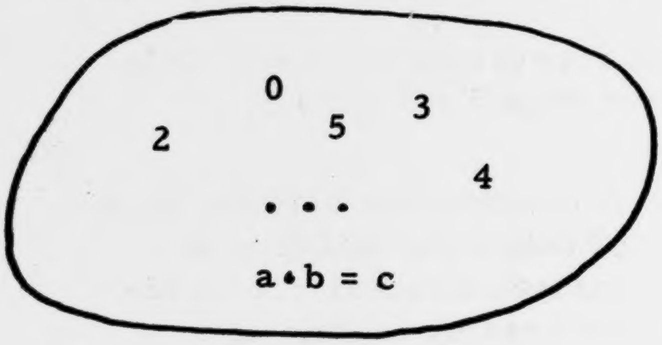
FUNDAMENTAL THEOREM OF ARITHMETIC:

Every composite number can be expressed as a product of primes which is unique except for the order of arrangement of the factors. This is called the "Fundamental Theorem of Arithmetic" or the "Unique Factorization Theorem".

CLOSURE AND UNIQUENESS

The assumptions for the operation of multiplication on elements of the set of natural numbers parallel those for the operation of addition.

It will be assumed that the product of any two numbers in the set N = {0, 1, 2, 3,•••} is a number in the given set N and that this product is unique. For example, 3 x 4 = 12 and 12 is also in N. Furthermore, since 3 x 4 equals 12 and only 12, the product is unique.



When the binary operation of multiplication is applied to numbers in the set N = {0, 1, 2, 3,•••}, the products are unique numbers of the same set. For this reason, the set N is said to be closed under the operation of multiplication.

The principles of closure and uniqueness for multiplication for the set N under the operation of multiplication may be stated:

$$a \cdot b = c$$

where a, b, and c are numbers in N and where c is uniquely determined by a and b.

COMMUTATIVE PRINCIPLE OF MULTIPLICATION

The order of multiplication of two numbers (factors) in the set N does not affect the product. For example 3 x 4 = 4 x 3 since 3 x 4 = 12

and 4 x 3 = 12. This illustrates the commutative principle of multiplication which may be stated:

$$a\cdot b = b\cdot a$$

for every pair of numbers a and b in set N.

Arrays may be used to help children discover the commutative principle of multiplication. A 6 x 4 array will have the same number of elements as a 4 x 6 array but will present a different appearance.

X X X X
X X X X
X X X X
X X X X
X X X X
X X X X

6 x 4

X X X X X X
X X X X X X
X X X X X X
X X X X X X

4 x 6

Examine again the multiplication table given earlier. How does it show the commutative principle?

ASSOCIATIVE PRINCIPLE OF MULTIPLICATION

Multiplication, like addition, is a binary operation. When more than two factors such as $2\cdot3\cdot4$ are to be multiplied, the factors are grouped as binary operations as $(2\cdot 3)\cdot 4$ or $2\cdot(3\cdot 4)$. The expression $(2\cdot 3)\cdot 4$ indicates 6 x 4 while $2\cdot(3\cdot 4)$ indicates $2\cdot 12$. Both 6 x 4 and 2 x 12 equal 24.

The associative principle of multiplication may be stated:

$$(a\cdot b\cdot)\cdot c = a\cdot(b\cdot c)$$

where a, b, and c are numbers in the set N.

EXISTENCE OF IDENTITY ELEMENT FOR MULTIPLICATION

The product of one and any number a is a. For example, 1 x 4 = 4; 1 x 5 = 5; 1 x 0 = 0. Since $1\cdot a = a$ for all a, 1 is called the identity element for multiplication or the multiplicative identity. Thus one has a similar role in multiplication to that of zero in addition.

The existence of an identity element for multiplication in the set N may be assumed:

$$1\cdot a = a\cdot 1 = a$$

where 1 and a are numbers in set N.

DISTRIBUTIVE PRINCIPLE OF MULTIPLICATION OVER ADDITION

Suppose that a man sells 3 books for \$2 each and then later in the day sells 4 more books for \$2 each. He receives \$14 whether he figures his sales as 7 books at \$2 each or as two separate sales of \$6 and \$8. Note that (3+4)2 = 2(3+4) = (2x3) + (2x4).

Since 2(3+4) = 2·3 + 2·4, the 2 is distributed to the 3 and 4. This can be illustrated by arrays:

```
          7                  3            4
     X X X X X X X         X X X        X X X X
2    X X X X X X X   = 2   X X X  + 2   X X X X
```

This illustrates the distributive principle that links the operations of multiplication and addition. The importance of this principle cannot be overemphasized.

That multiplication is distributive over addition in the set N may be stated:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

where a, b, and c are numbers in the set N.

USING THE DISTRIBUTIVE PRINCIPLE

Children can discover easier combinations and verify their results by using the distributive principle as shown in the following example:

6 x 9 = 6 x (5+4) = (6x5) + (6x4)

```
           9                        5              4
  x x x x x x x x x            x x x x x        x x x x
  x x x x x x x x x            x x x x x        x x x x
6 x x x x x x x x x   =   6    x x x x x   + 6  x x x x
  x x x x x x x x x            x x x x x        x x x x
  x x x x x x x x x            x x x x x        x x x x
  x x x x x x x x x            x x x x x        x x x x
```

MULTIPLICATION OF LARGER NUMBERS:

The principle that multiplication is distributive over addition is basic to the familar rule for multiplying larger numbers. In the following examples compare the standard process with the process showing a vertical and horizontal step-by-step application of the distributive principle.

$$\begin{array}{r} 165 \\ \times\ 5 \\ \hline 825 \end{array}$$

$$\begin{array}{rcr} & & 165 \\ & & \times\ 5 \\ \cline{3-3} 5 \times 5 & = & 25 \\ 5 \times 60 & = & 300 \\ 5 \times 100 & = & 500 \\ \cline{3-3} 5 \times (100+60+5) & = & 825 \end{array}$$

$$\begin{aligned} 5 \times 165 &= 5 \times (100+60+5) \\ &= 5\times100 + 5\times60 + 5\times5 \\ &= 500 + 300 + 25 \\ &= 825 \end{aligned}$$

$$\begin{array}{r} 23 \\ \times\ 45 \\ \hline 115 \end{array}$$

$$\begin{array}{l} \begin{array}{r} 23 \\ \times 45 \\ \hline \end{array} \\ \left.\begin{array}{r@{\;=\;}l} 15 & 5 \times 3 \\ 100 & 5 \times 20 \end{array}\right\} 5(20+3) \\ \left.\begin{array}{r@{\;=\;}l} 120 & 40 \times 3 \\ 800 & 40 \times 20 \end{array}\right\} 40(20+3) \\ \hline 1035 = 5 \times (20+3) + 40\times(20+3) \end{array}$$

$$\begin{aligned} 45 \times 23 &= (40+5)23 \\ &= 40(23) + 5(23) \\ &= 40(20+3) + 5(20+3) \\ &= 800 + 120 + 100 + 15 \\ &= 1035 \end{aligned}$$

SUMMARY OF PRINCIPLES FOR THE NATURAL NUMBERS

For all numbers a, b, c, ... in the set $N = \{0, 1, 2, 3, \cdots\}$

	Addition	Multiplication
Closure and uniqueness	$a + b = c$	$a \cdot b = c$
Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Identity Element	$a + 0 = a$	$1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
Distributivity	$a \cdot (b + c) = ab + ac$	

REFERENCES FOR TEACHERS UNIT IV

Adler--Magic House of Numbers: pp. 7-24, 43-51, 53-59

Adler--Mathematics, Exploring the World of Numbers & Space:
pp. 16-17, 30-32, 81

Adler--The New Mathematics: pp. 17-27

Banks--Elements of Mathematics: pp. 85-94, 100-107

C-2 Bulletin, Thinking in the Language of Mathematics: pp. 7-10, 43-55

Dantzig--Number, The Language of Science: pp. 58-64, 273-274

Heath--Mathemagic: pp. 34-50, 61-63, 87-98, 120-123

Hogben--The Wonderful World of Mathematics: pp. 44-51

Johnson & Glenn--Computing Devices: pp. 11-14

Larsen--Magic Squares, Circles and Stars: pp. 3-15

Larsen--Magic Squares: pp. 23-31

Larsen--Prime Numbers: pp. 1-15

Merrill--Excursions in Mathematics: pp. 67-76

National Council, 24th Yearbook--The Growth of Mathematical Ideas: pp. 20-29

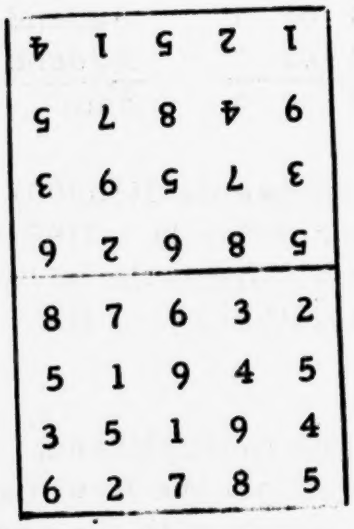
National Council, 25th Yearbook--Instruction in Arithmetic: pp. 33-54

Osborn, DeVault, Boyd, and Houston--Extending Mathematics Understanding:
pp. 25-50

Smith and Ginsburg--Numbers and Numerals: pp. 25-34

Swain--Understanding Arithmetic: pp. 46-58, 67-73, 80-88, 113-129

4.01 Number Checkers



Number checkers can be used to provide interesting drill on the addition facts. The game is played on a 7 1/2" x 12" board which may be made of cardboard. Each player will need a number of numerals equal to twice the sum being studied. If the addition facts of 10 are being studied, each student will need 20 numerals of two 1's, two 9's, two 2's, two 8's, two 3's, two 7's, two 4's, two 6's, but four 5's since two sets of 5 + 5 would call for four 5's.

If the addition facts of 8 are being studied, each student will need 16 numerals consisting of two 1's, two 7's, two 2's, two 6's, two 3's, two 5's, and four 4's. However, to introduce 8 + 0 = 8 use the same board as the one for 10 with two 8's and two 0's.

By agreement each child may arrange on the board either his own numerals any way he wishes or those of his opponent. A move may be made upward, downward, sideways, or diagonally, and only one square at a time. The purpose is to place one numeral on another so that the numbers they represent total the agreed upon sum, e.g. 10. The cards with numerals representing numbers that total 10 are removed from the board. If a numeral cannot be placed on another to represent a total of 10, it may be moved to an adjoining blank space in preparation for a later move. The first person getting ten totals of 10 is the winner.

4.02 Casting Out Nines

According to legend, a primitive chief once recorded the number of sheep in his fold by placing pebbles on the ground as his sheep passed by. When he had placed 10 pebbles in a spot corresponding to the ones place, he took the 10 pebbles and placed one in the tens place and put the other nine aside or cast out nine. Casting out nines may be used to check the four operations.

This is an easy check to use on an abacus and may be applied to any of the fundamental operations. To check addition, add the numbers represented by the digits of each addend and the sum. Divide each by 9 and record the remainder. Add the remainders for the addends and divide the result by 9. If this result is greater than 9, again divide by 9 and record the remainder. This remainder should equal the remainder for the sum.

Study the example below:

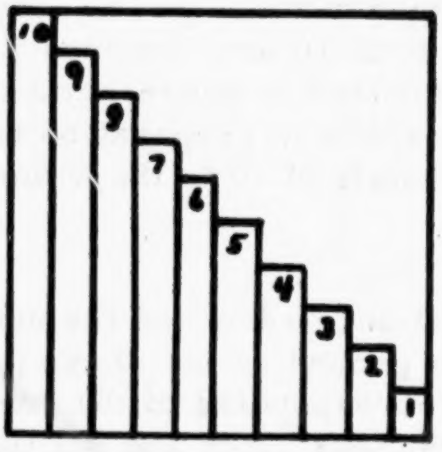
32716	3 + 2 + 7 + 1 + 6 = 19 or 1	Addend
+ 52934	5 + 2 + 9 + 3 + 4 = 23 or 5	Addend
85650	8 + 5 + 6 + 5 + 0 = 24 or 6	Sum

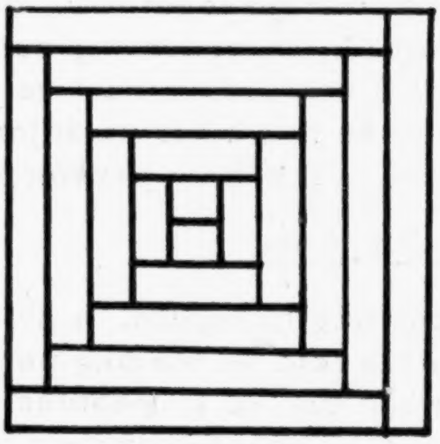
Why can this check be used? 32716 represents 3(10000) + 2(1000) + 7(100) + 1(10) + 6 or 3(9999 + 1) + 2(999 + 1) + 7(99 +1) + 1(9 + 1) + 6. Now 9999, 999, 99 and 9 are all divisible by 9. Therefore, if 3·1 + 2·1 + 7·1 + 1·1 + 1·6 is divisible by 9, the original number is divisible by 9.

To check multiplication, cast out 9's in the multiplicand, in the multiplier and in the product. Then multiply the first two results and cast out 9's. The answer should agree with the result for the product.

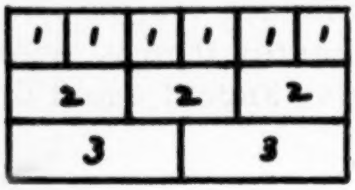
4.03 The 10-Frame

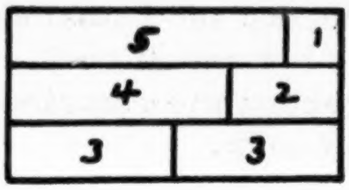
A device known by various names but perhaps most frequently called the 10-Frame is available at game and variety stores. It may be constructed using cardboard or wood. Strips representing unit lengths of 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 may be used in kindergarten to cover a 10 x 10 square. Some patterns that may result include the following:





In the primary grades strips may be used to help children discover the relationship between various lengths when introducing relative size. They may also be used in developing new concepts in intermediate grades when studying fractions.

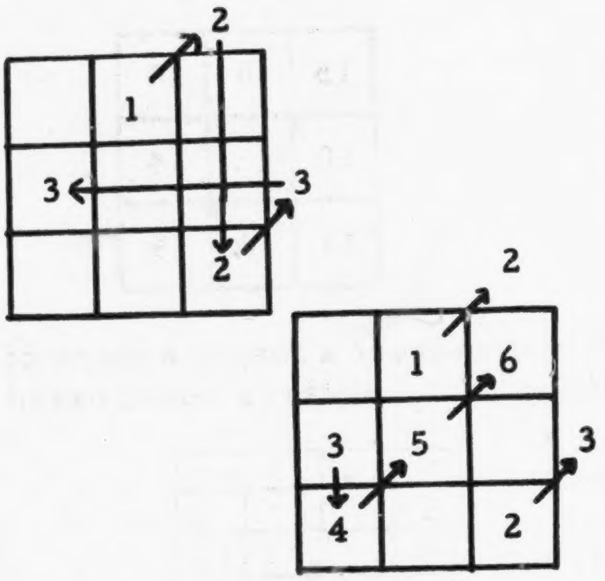




4.04 Magic Squares

Magic squares are actually very unmagical. They are squares whose sums (or products) in all rows, columns, and diagonals are the same. The ancients called these magic squares because they thought such squares brought good health and luck. Albrecht Durer drew a magic square in his painting "Melencolia".

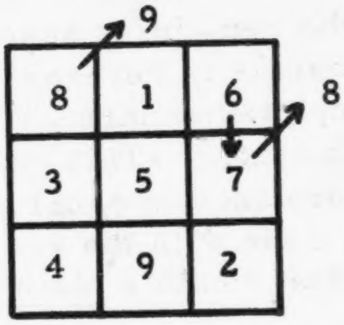
A pattern for setting up a magic square with an odd number of rows and columns follows:



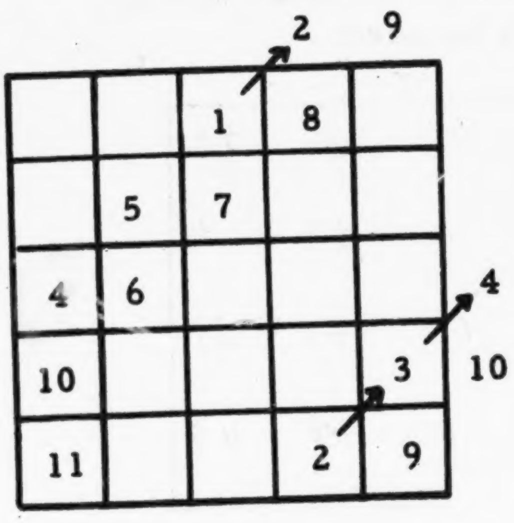
Place the first numeral in the middle square of the top row. Then always move diagonally to the upper right as shown by the arrows. Since no square is diagonally opposite to '1', write '2' in the lowest square in the column. Next move diagonally from '2'. No square is diagonally opposite '2', and '3' is placed in the first square in the second row. Since '1' fills the square to the upper right of '3', then '4' is written below '3'.

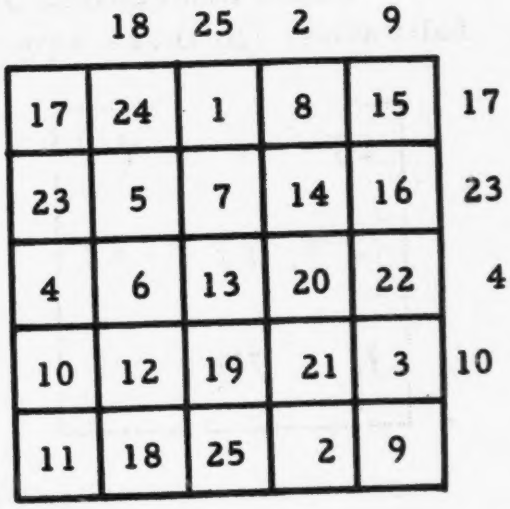
To finish the square, follow the arrows in the diagrams. Notice that '7' is below '6'.

This same pattern may be used to create a 5 x 5 square, 7 x 7 square, or any square with an odd number of rows and columns.



The construction of a 5 x 5 magic square is shown below.





The total can be made to equal any sum if fractions are permitted; but only natural numbers should be used to construct the first squares studied. A general pattern is shown at the right where the smallest number is represented by n and the largest by n + 8.

n + 7	n	n + 5
n + 2	n+4	n + 6
n + 3	n+8	n + 1

Let 3n + 12 = S, where S is any total you wish. Replace S with some value and solve for n. Suppose that the total for each row and and column is to be 36. Then 3n + 12 = 36, 3n = 24, n = 8.

Any set of consecutive numbers, of consecutive even numbers, etc., may be used.

15	8	13
10	12	14
11	16	9

The numerals 1-16 may be used to construct a magic square of order 4 by drawing an X in the square and filling in the squares not crossed by the X with the numerals that would be assigned to them if all the squares were filled in consecutively. Then with the numerals 1, 4, 6, 7, 10, 11, 13, and 16 which were omitted, fill in the remaining spaces using the numerals in reverse order. The completed square will be as shown at the right. This can also be generalized to equal any sum as was done with the square of order 3. Consult Heath's Mathemagic for other magic squares.

	2	3	
5			8
9			12
	14	15	

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

4.05 Prime Number Magic Squares

Some interesting prime number magic squares are the following: (In these squares 1 may also be used.)

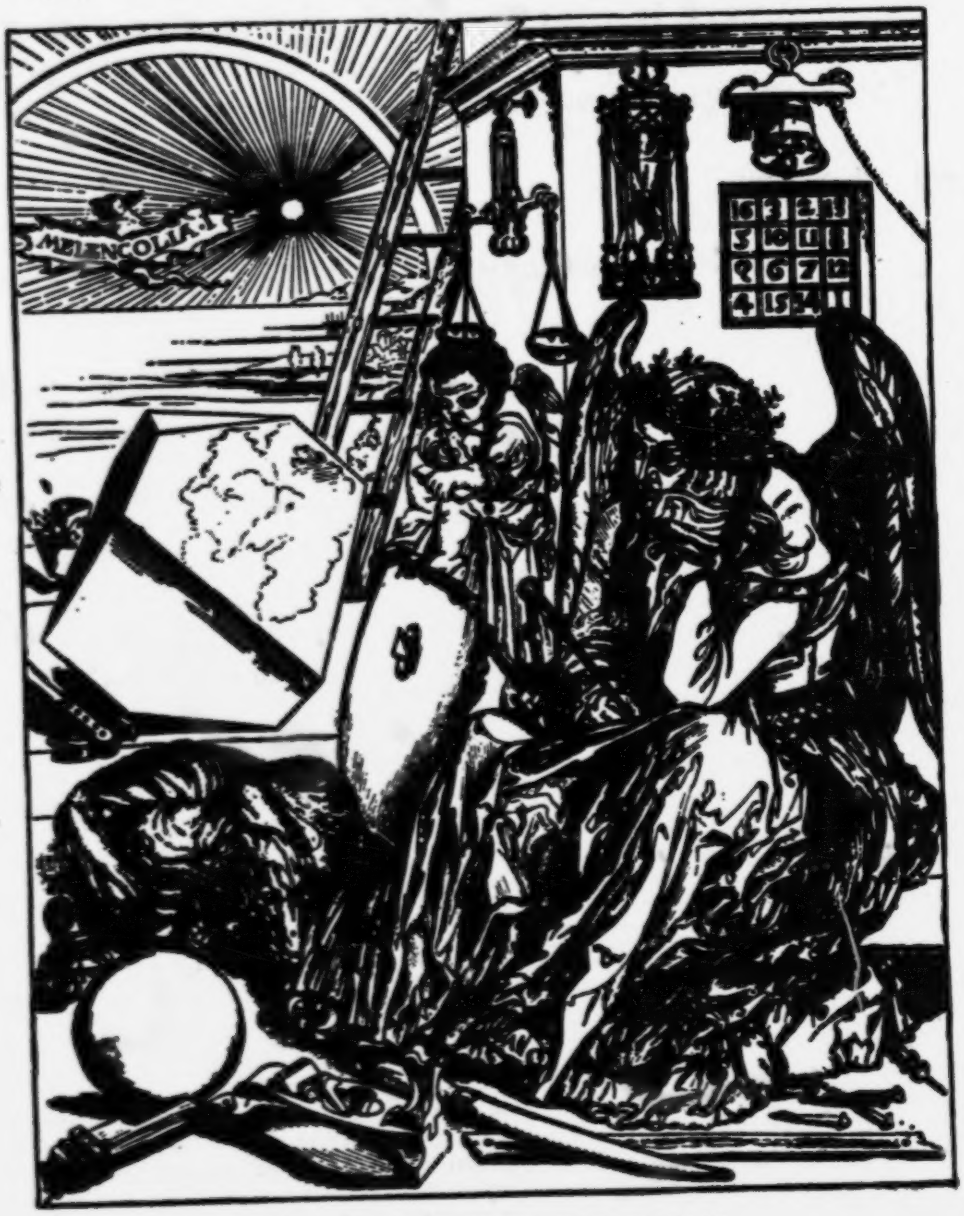
67	1	43
13	37	61
31	73	7

PRIME NUMBER MAGIC SQUARES

3	71	5	23
53	11	37	1
17	13	41	31
29	7	19	47

4.06 Magic Squares in Art

Magic squares of order 4 were supposed by Renaissance astrologers to combat melancholy which is portrayed by Albrecht Durer in his engraving Melencolia. Notice the disheveled figure of Melancholy, the unused tools, the empty scale pans, the half starved hound, and the cherub waiting for dictation while time is running out in the hourglass on the wall. Most people believe that Durer was trying to depict the sullen mood of a creative genius unable to act.



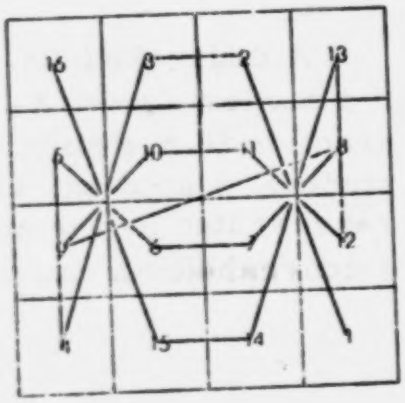
ALBRECHT DURER'S MELENCOLIA.

The magic square is reproduced below. Note that he interchanged the two middle columns so that the two middle cells of the bottom row would indicate the year he made the engraving.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

4.07 Patterns of Magic Squares

The "magic line" of Durer's square shown at the left is obtained by connecting the numerals in order. Very interesting designs often appear and may interest some of the students.



ACTIVITIES

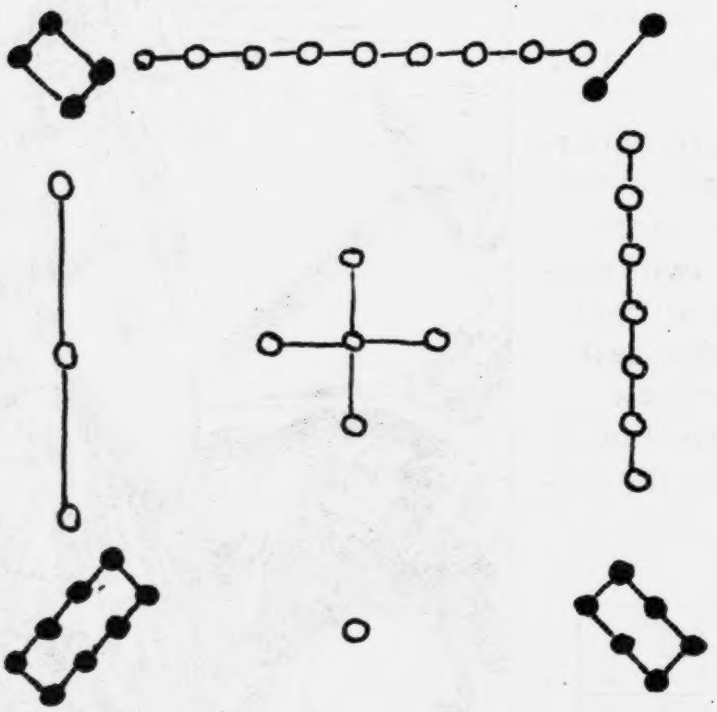
4.08 Magical and Unmagical Blocks

Blocks with some faces that are magic squares and some that are not magic squares can be used for addition practice. Students can be asked to find out how many of the faces are magic squares.

A very interesting block of this type with magic squares on all six faces and using the numbers 1-96 with no repetitions is given in Heath's Mathemagic on p. 122.

4.09 Lo-Shu

Read the history of Lo-Shu, the ancient magic square, shown below:



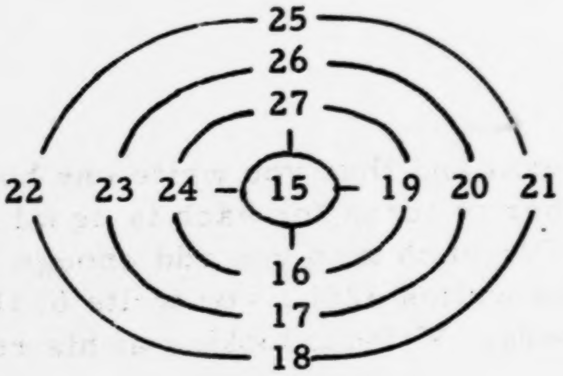
IN ONE OF THE MOST ANCIENT CHINESE BOOKS, THE I-KING, APPEARS WHAT IS PROBABLY THE OLDEST MAGIC SQUARE. THE BLACK CIRCLES REPRESENTED FEMININE OR EVEN NUMBERS AND THE WHITE ONES REPRESENTED MASCULINE OR ODD NUMBERS. THIS MAGIC SQUARE WAS CALLED THE LO-SHU

4.10 Dodecahedron

A dodecahedron, one of the five regular solids, has twelve faces. If a numeral from 1-12 is on each face, the dodecahedron can afford practice in multiplication. If the multiplication facts of six are being studied, a student can toss the dodecahedron and multiply the number represented by the numeral on the top face by 6. Other games using a dodecahedron can be invented.

4.11 Magic Circles

Concentric "magic" circles can be constructed so that the numbers represented on each circle total the same and also those represented on each diameter total the same. These can afford practice in addition.



$$25 + 26 + 27 + 15 + 16 + 17 + 18 = 22 + 23 + 24 + 15 + 19 + 20 + 21$$

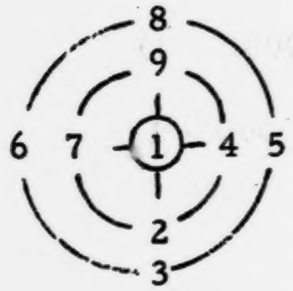
$$26 + 27 + 15 + 16 + 17 = 23 + 24 + 15 + 19 + 20$$

$$24 + 15 + 19 = 27 + 15 + 16$$

$$24 + 27 + 19 + 16 = 23 + 26 + 20 + 17$$

$$= 22 + 25 + 21 + 18$$

Diagrams with more or fewer circles may be constructed with numbers suitable for the grade level. An even number of diameters is drawn--2, 4, 6, 8, etc. Starting in the center, number outward on half of the diameters and inward on the other half. A pattern is shown below. Typical addition problems are also shown.



$$\begin{array}{r} 8\\ 9\\ 1\\ 2\\ 3\\ \hline 23 \end{array} \quad \begin{array}{r} 6\\ 7\\ 1\\ 4\\ 5\\ \hline 23 \end{array} \quad \begin{array}{r} 7\\ 2\\ 4\\ 9\\ \hline 22 \end{array} \quad \begin{array}{r} 6\\ 3\\ 5\\ 8\\ \hline 22 \end{array} \quad \begin{array}{r} 9\\ 1\\ 2\\ \hline 12 \end{array} \quad \begin{array}{r} 7\\ 1\\ 4\\ \hline 12 \end{array} \quad \begin{array}{r} 6\\ 1\\ 5\\ \hline 12 \end{array} \quad \begin{array}{r} 8\\ 1\\ 3\\ \hline 12 \end{array}$$

4.12 Tic-Tac-Toe with Addition or Multiplication Facts

+	3	6	9
8		14	⑰
5		11	14
4		⑩	⑬

x	3	4	5
6			
7			
8			

Divide the class into teams. Let a child on each team name a sum (or product) such as 5 + 6 = 11. Enter the result in the proper square. Play by the usual rules for Tic-Tac-Toe. If a child makes a mistake no entry is made and his team loses its turn.

4.13 Nine's Game

Choose any 5-digit number, e.g. 31465. Add a number represented by the first digit to the original number and write the numeral representing the last 4 digits of this sum.

$$\text{Example:}\quad \begin{array}{r} 31465 \\ 3 \\ \hline 31468 \end{array} \quad \text{Record 1468.}$$

Let a student write any 4-digit numeral and then you write one below it. Continue to take turns until the number of turns for each is equal to the first digit of the original numeral. For each turn you add enough to his number to make a total of 9999. If he writes 3268, you write 6731. Finally ask the student to total the numbers. Without looking at his result, tell him the sum.

The sum will be the same as the original number. This can be seen from the illustration below:

$$\begin{array}{rrl} & 1468 & = N \\ \text{A's choice} & 3268 & \\ \text{your \quad "} & 6731 & (10000 - 1) \\ \text{A's choice} & 1582 & \\ \text{your \quad "} & 8417 & (10000 - 1) \\ \text{A's choice} & 3742 & \\ \text{your \quad "} & 6257 & (10000 - 1) \\ \hline & 31465 & = N \end{array}$$

4.14 Cryptarithmetic

In place of numerals many interesting puzzles use letters which students are to decipher. For this reason such problems are often termed cryptarithmetic puzzles. They afford entertainment and practice in the four operations. Problems a and b below are addition problems and problem c is a multiplication problem.

$$\text{a.}\ \begin{array}{ccccc} & S & E & N & D \\ & M & O & R & E \\ \hline M & O & N & E & Y \end{array} \qquad \text{b.}\ \begin{array}{ccccc} & S & E & N & D \\ & M & O & R & E \\ & G & O & L & D \\ \hline M & O & N & E & Y \end{array} \qquad \text{c.}\ \begin{array}{ccccc} & & F & U & R \\ & \times & D & O & G \\ \hline & A & G & E & R \\ D & R & I & P & \\ F & U & R & & \\ \hline R & I & P & E & R \end{array}$$

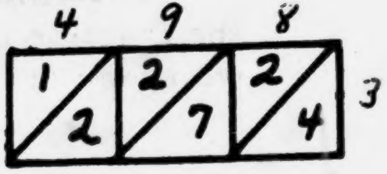
One solution for each problem follows:

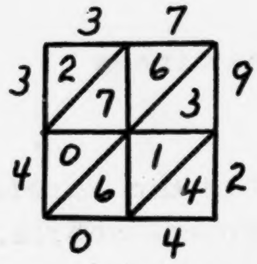
$$\text{a.}\ \begin{array}{ccccc} & 9 & 5 & 6 & 7 \\ & 1 & 0 & 8 & 5 \\ \hline 1 & 0 & 6 & 5 & 2 \end{array} \qquad \text{b.}\ \begin{array}{ccccc} & 4 & 6 & 7 & 3 \\ & 1 & 5 & 0 & 6 \\ & 9 & 5 & 8 & 3 \\ \hline 1 & 5 & 7 & 6 & 2 \end{array} \qquad \text{c.}\ \begin{array}{ccccc} & & 3 & 9 & 5 \\ & & 1 & 4 & 7 \\ \hline & 2 & 7 & 6 & 5 \\ 1 & 5 & 8 & 0 & \\ 3 & 9 & 5 & & \\ \hline 5 & 8 & 0 & 6 & 5 \end{array}$$

4.15 Lattice Multiplication

Lattice multiplication was used in the early Middle Ages. A lattice is a framework of crossed strips resembling a window grating (gelosia in Italian and jalosie in French), and for this reason this method of multiplication is often called the grating or gelosia method.

Rule a rectangle into squares with a diagonal to divide each square. Write the multiplicand at the top of the lattice and the multiplier at the right.





Each product is written in a square with the ones in the lower part and the tens in the upper part. To find the product add the numbers represented in each diagonal strip. In the illustration the product of 37 and 92 is represented by the numerals at the left and along the bottom of the lattice. 92 x 37 = 3404.

4.16 Fibonacci Numbers

Choose any two one-digit numbers. Then add the last one with the preceding number until 10 numbers make up the sequence. Add. The total will be equal to the product of 11 times the fourth number from the bottom.

	1	a
	7	b
	8	a + b
	15	a + 2b
	23	2a + 3b
	38	3a + 5b
	61	5a + 8b
	99	8a + 13b
	160	13a + 21b
	259	21a + 34b
11 x 61 =	671	55a + 88b

which divided by 11 equals 5a + 8b which is the 4th number from the bottom.

The Fibonacci numbers are quite well known in mathematics. In 1202, Leonardo of Pisa, nicknamed Fibonacci came across his celebrated sequence of integers in connection with the breeding of rabbits. He assumed that rabbits live forever, and that every month each pair begets a new pair which becomes productive at the age of two months.

In the first month the experiment begins with a newborn pair of rabbits. In the second month, there is still just one pair. In the third month there are two; in the fourth, three, etc. Let P_n denote the number of pairs of rabbits in the nth month. Then,

n:	1	2	3	4	5	6	7	8	9	...
P_n	1	1	2	3	5	8	13	21	34	...

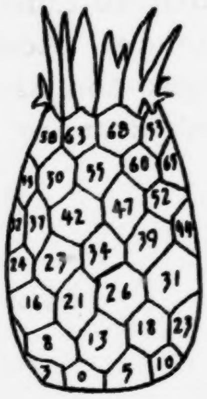
giving us the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, etc.

Four centuries later, Girard noticed that each of these numbers (after the second) is equal to the sum of the preceding two.

The ratios of alternate Fibonacci numbers are said to measure the fraction of a turn between successive leaves along a twig, and we speak of "1/2 phyllotaxis" or "1/3 phyllotaxis".

1:2	grasses
1:3	sedges
2:5	apple and cherry trees
3:8	common plantain, poplar, and pear trees
5:13	leek, willow, and almond trees

Another case of phyllotaxis is the arrangement of the florets of a sunflower, or of the scales of a pine cone, in spiral whorls. The numbers of the right-handed and left-handed whorls are two consecutive Fibonacci numbers, such as 2 and 3 for the balsam cone, 3 and 5 for the hemlock cone, 5 and 8 for the pine cone, 8 and 13 for the pineapple (clearest example of all), and larger numbers for sunflowers such as 34 and 55, and a giant sunflower 55 and 89. The Russians claim a super-giant 89 and 144.



FIBONACCI NUMBERS AS EXHIBITED BY THE PINEAPPLE

Exercises based upon Fibonacci series are easy to construct and they provide interesting discovery exercises for children. Make up a series and omit some of the numbers in the series.

2 3 5 8 ___ ___ 34 ___ 89

1 3 4 ___ ___ ___ 29 47 ___

4.17 Interesting Products

Select a number from U = {1, 2, 3, 4, 5, 6, 7, 9} and multiply it by 9. Now multiply 12345679 by this product.

Example: 9 x 3 = 27

$$\begin{array}{r} 12345679 \\ 27 \\ \hline 86419753 \\ 24691358 \\ \hline 333333333 \end{array}$$

ACTIVITIES

4.18 An Addition Matrix

		3
1	2	0
		4

Interesting drill in addition can be provided by a matrix where the sums of numbers represented in different rows and columns will always be the same. When the matrix on the left is completed, the sums will be 10. Note that 1 + 2 + 3 + 0 + 4 = 10 and that this determines the sum 10. The top row is filled in by adding 3 to each of the first two numbers represented in the middle row. The bottom row is filled in by adding 4 to each of the first two numbers in the middle row.

4	5	3
1	2	0
5	6	4

Example: 1(first column, second row) + 6(second column, third row) + 3 (third column, first row) = 10.

This activity can be adapted for any grade level and may include fractions and decimals. For example, a matrix with a sum of 1963 for five numbers each represented in a different row and column may be constructed by first making the sum of the numbers represented in the middle row and last column total 1963.

In the 5 x 5 array below if one numeral is selected from each column and each row, the total sum represented by the five numerals will be 1963. For example, if 508 is selected from the first column and first row, 567 from the second column fifth row, 115 from the third column and third row, 536 from the fourth column and second row, and 237 from the fifth column and fourth row, then 508 + 567 + 115 + 536 + 237 = 1963.

508	355	258	364	143
680	527	430	536	315
365	212	115	221	0
602	449	352	458	237
720	567	470	576	355

The sum 1963 may be found $5^2 \times 4^2 \times 3^2 \times 2^2 \times 1^2$ different ways from this array.

The middle row and last column were first made to total 1963. The first four numbers represented in the top row were found by adding each of the numbers represented by numerals in the third row to the number represented by the first numeral in column five. Row two was constructed by adding 315 to each number represented in the third row, row four by adding 237, and row five by adding 355.

4.19 ACTIVITIES

Famous Mathematicians

1473-1543	Nicholas Copernicus was a famous Polish astronomer as well as a mathematician.
1510-1558	Robert Recorde, the author of an important arithmetic textbook published in England, invented the symbol "=".
1540-1603	Francois Vieta, a French mathematician, contributed extensively to the development of algebra and trigonometry. He was among the first to use letters to represent numbers in algebra and often used vowels for unknowns and consonants for knowns.
1550-1617	John Napier, a wealthy Scotsman, invented Naperian logarithms which use a base e. According to many stories, he was regarded as a magician. Read about how he discovered which one of his servants had been cheating him by using a black rooster. In Unit II a set of Napier's Rods was described.
1561-1631	Henry Briggs, an English mathematician, is remembered as the father of common logarithms which use the base 10.
1564-1642	Galileo Galilei, the son of a Florentine nobleman, was interested in mathematics and music but also studied medicine. He was tried for heresy because of his theory that the earth moved around the sun.
1571-1630	Johann Kepler was a famous German astronomer whose research helped number our years by fixing the approximate year of the birth of Christ as 8 B. C.
1574-1660	William Oughtred, an English mathematician, is often called the father of the slide rule.
1588-1648	Marin Mersenne was a friar and a lifelong friend of Descartes. Find out about Mersenne numbers.
1596-1650	Rene Descartes, a famous French mathematician, is often referred to as the father of analytic geometry. Cartesian coordinates are named for him. His many adventures make interesting reading.
1598 - 1647	Bonaventura Cavalieri, a pupil of Galileo, did work in mathematics which was a precursor of the integral calculus.

UNIT IV

EXERCISES

1. Let $A = \{△, □\}$ and $B = \{▽, △, ⊠\}$
   - a. Represent $A \cup B$
   - b. $n(A)$
   - c. $n(B)$
   - d. $n(A) + n(B)$

2. Is the commutative property valid for the following operations:
   - a. Opening the door--going outside?
   - b. Putting on your shoes--putting on your socks?

3. Using the lattice method of multiplication, multiply the following:

   a. $\begin{array}{r} 37 \\ \times\ 85 \\ \hline \end{array}$ b. $\begin{array}{r} 283 \\ \times\ 79 \\ \hline \end{array}$

4. Solve the following and check by casting out nines:

   a. $\begin{array}{r} 942 \\ 769 \\ 146 \\ \hline \end{array}$ b. $\begin{array}{r} 287 \\ \times\ 36 \\ \hline \end{array}$

5. (a) How does 9 compare with 10?

   (b) What similar check can be used in base 8?

6. Will checks of this type always detect an error?

7. Justify each step given below:

   a.
$$\begin{aligned} 8 + 9 &= 8 + (2 + 7) \\ &= (8 + 2) + 7 \\ &= 10 + 7 \\ &= 17 \end{aligned}$$

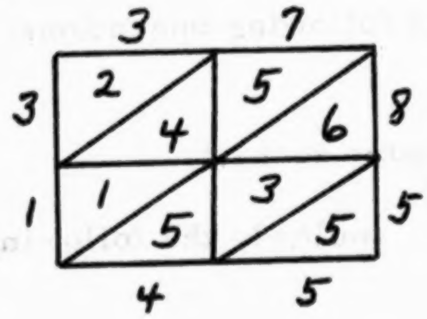
   b.
$$\begin{aligned} 49 \times 73 + 27 \times 49 &= 49 \times 73 + 49 \times 27 \\ &= 49\ (73 + 27) \\ &= 49\ (100) \\ &= 4900 \end{aligned}$$

EXERCISES (answers) UNIT IV

1. a. { △, □, ▽, ⏢, ⊠ }
   b. 2
   c. 3
   d. 5

2. a. No
   b. No

3. a.



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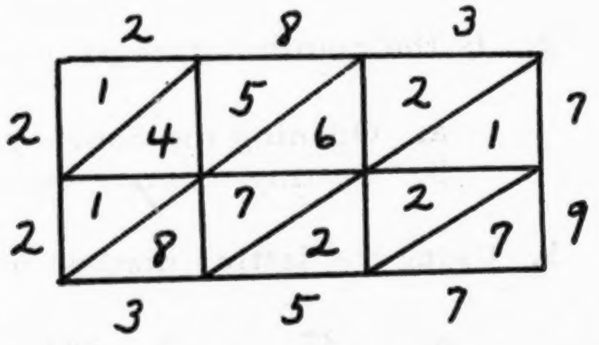
b.



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4. a.

	9's
942	6
769	4
146	2
1857	3

b.

	9's
287	8
x 36	x 0
1722	0
861	
10332	

5. a. 9 is 1 less than 10

   b. Cast out 7's in base 8

6. No Transposition is not detected by casting out 9's and 2 errors may offset one another.

7. a. Another name for 9
   Associative principle for addition
   Binary operation of addition
   Binary operation of addition

   b. Commutative principle for multiplication
   Distributive principle of multiplication over addition
   Binary operation of addition
   Binary operation of multiplication

ASSIGNMENT UNIT IV

1. Sketch pictures of an abacus showing:

   a. 34 c. 2 tens

   b. 902 d. 4 tens and 13 ones

2. Let A = {Kay} and B = {Helen, Ann}. Represent the following:

   a. $A \cup B$
   b. $n(A \cup B)$
   c. $n(A)$
   d. $n(B)$
   e. $n(A) + n(B)$

3. It has been assumed that the set of all natural numbers is closed under addition and multiplication. Which of the following sets are closed?

   a. the even numbers under addition
   b. the odd numbers under addition
   c. the even numbers under multiplication
   d. the odd numbers under multiplication

4. In which of the following examples is the commutative property valid:

   a. 6 + 3 = 3 + 6 ?
   b. Taking a shower and removing your clothes = removing your clothes and taking a shower ?
   c. Brushing your teeth and combing your hair = combing your hair and brushing your teeth ?

5. Add the following and check by casting out nines:

   a. 4321
   6895
   4
   32942

   b. 46281
   3459
   81454
   41927

6. In the addition of ▭ + △ = ⌂, is

   a. ⌂ doubled if ▭ is doubled ?
   b. ⌂ doubled if ▭ and △ are both doubled?
   c. △ doubled if ⌂ is doubled and ▭ remains unchanged?

7. Use the properties of addition and multiplication to explain the products: (a) 6 x 53; (b) 42 x 75.

8. Consider the multiplication of $\square \times \triangle = \nabla$

   a. What happens to the product if $\triangle$ is doubled?
   b. What happens to the product if $\square$ is halved?
   c. What happens to the product if $\triangle$ is doubled and $\square$ is halved?
   d. How much larger will the product be if $\triangle$ is increased by 2?
   e. What will the result be if $\square$ or $\triangle$ is 0?
   f. Find the value of $\square$ if $\triangle = \nabla$

9. Addition and multiplication are both binary operations. A pair of numbers may be combined in a definite order using one of these two basic operations. Complete the following table.

Mathematical Sentence	Number Pair	Binary Operation	Truth Set
a. 7 + 2 = N	7, 2	+	{9}
b. 7 x 2 = N	7, 2	x	{14}
c. 6 + 2 = N	______	______	______
d. 8 x 3 = N	______	______	______
e. ______	6, 7	______	{42}
f. ______	8, 9	x	______
g. ______	9, 7	______	{16}
h. ______	5, 8	______	{40}

10. Given U = {0, 1, 2, 3, 4, 5, . . . }, solve the following:

   a. $3 + \square = 8$
   b. $3 + \square > 8$
   c. $3 + \square < 8$
   d. $3 + \square \neq 8$
   e. $3 + \square < 4$
   f. $3 + \square < 3$
   g. $2 \times \square = 6$
   h. $2 \times \square > 6$
   i. $2 \times \square \leqslant 6$
   j. $\square + 4 \geqslant 16$

JUST FOR FUN

Exercises 11-12 are just for fun. Helpful hints will be found in the ACTIVITIES section.

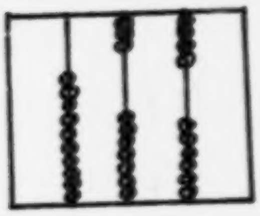
11. Construct a magic square of order 4 using the numerals 5-20 Trace a pattern by joining consecutive even numerals; by joining consecutive odd numerals.

12. Use the lattice method to multiply the following:

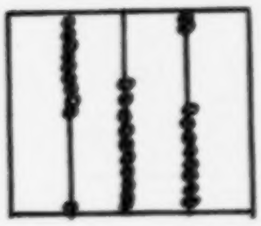
   a. $\begin{array}{r} 48 \\ \underline{\times\ 34} \end{array}$ b. $\begin{array}{r} 28 \\ \underline{\times\ 192} \end{array}$ c. $\begin{array}{r} 418 \\ \underline{\times\ 287} \end{array}$

ASSIGNMENT (answers)

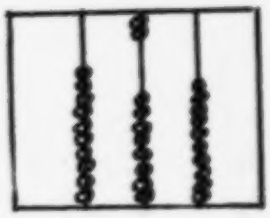
1. a.



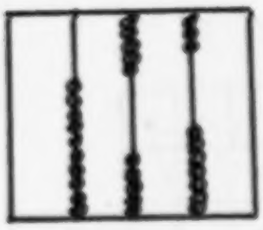
b.



c.



d.



2. a. {Kay, Helen, Ann}
   b. 3
   c. 1
   d. 2
   e. 3

3. a. yes
   b. no
   c. yes
   d. yes

4. a and c.

5. a.

	9's
4321	1
6895	1
4	4
32942	2
44162	8

b.

	9's
46281	3
3459	3
81454	4
41927	5
173121	6

6. a. No
   b. Yes
   c. No

7. a. $6 \times 53 = 6 \times (50 + 3)$
$= 6 \times 50 + 6 \times 3$
$= 300 + 18$
$= 318$

b. $42 \times 75 = 42 \times (70 + 5)$
$= 42 \times 70 + 42 \times 5$
$= 70 \times 42 + 5 \times 42$
$= 70 \times (40 + 2) + 5 \times (40 + 2)$
$= 70 \times 40 + 70 \times 2 + 5 \times 40 + 5 \times 2$
$= 70 \times 40 + 5 \times 40 + 70 \times 2 + 5 \times 2$
$= (2800 + 200) + (140 + 10)$
$= 3000 + 150$
$= 3150$

8. a. Doubled
   b. Halved
   c. Remains the same
   d. 2□
   e. 0
   f. 1

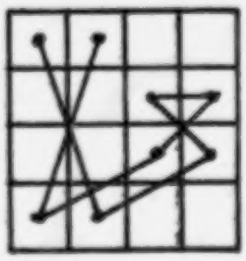
ASSIGNMENT (answers) **UNIT IV**

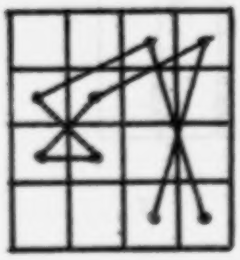
9.	c. ______	6, 2	+	{8}
	d. ______	8, 3	x	{24}
	e. 6 x 7 = N	______	x	
	f. 8 x 9 = N	______	______	{72}
	g. 9 + 7 = N	______	+	______
	h. 5 x 8 = N	______	x	______

10. a. {5}
b. {6, 7, 8, ···}
c. {0, 1, 2, 3, 4}
d. {0, 1, 2, 3, 4, 6, 7, 8, ···}
e. {0}
f. { }
g. {3}
h. {4, 5, 6, ···}
i. {0, 1, 2, 3}
j. {12, 13, 14, ···}

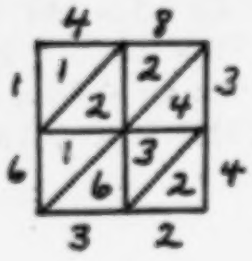
11.

20	6	7	17
9	15	14	12
13	11	10	16
8	18	19	5

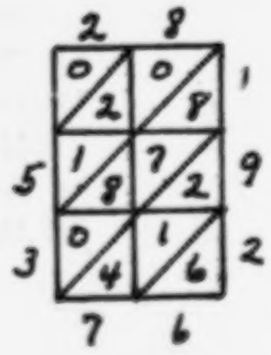




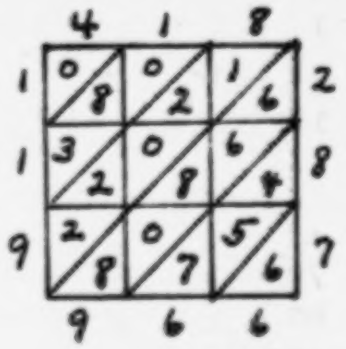
12.


4 8
1 2 2 4
1 6 3 2
1 6 3 4
3 2

a. 1632


2 8
0 2 0 8
1 8 7 2
0 4 1 6
1 5 3 9 2
7 6

b. 5376


4 1 8
0 8 0 2 1 6
3 2 0 8 6 4
2 8 0 7 5 6
1 1 9 2 8 7
9 6 6

c. 119966

UNIT V

OPERATION OF SUBTRACTION

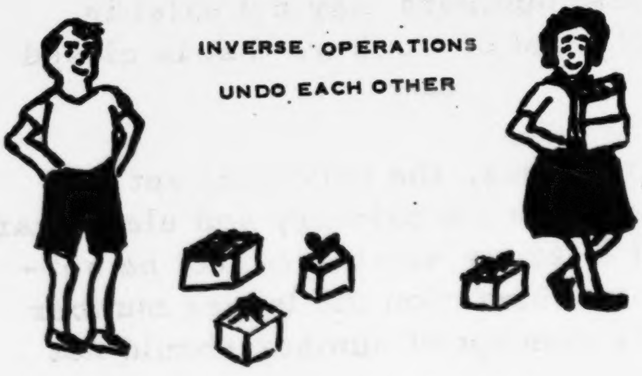
UNIT V. OPERATION OF SUBTRACTION

> *"When I use a word, "Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean--*
> *Neither more nor less."*
> *"The question is," said Alice, "whether you can make words mean so many different things."*
>
> *-----Lewis Carroll*

INTRODUCTION

In Unit IV principles of addition and multiplication for the set N = {0, 1, 2, 3, ••• } were presented. In this unit, the operation of subtraction will be discussed. The integers, a set, constructed from the set of natural numbers, will be introduced.

INVERSE OPERATIONS



Two operations are called inverse operations if when performed in succession they annul or undo each other. The operation of subtraction is the inverse operation of addition, and addition is the inverse operation of subtraction. If 6 + 5 = 11, then 11 - 5 = 6 and 11 - 6 = 5. Likewise the operation of multiplication is the inverse operation of division and division is the inverse operation of multiplication. If 10 ÷ 5 = 2, then 2 x 5 = 10. This unit will be confined principally to the operation of subtraction as the inverse operation of addition while division as the inverse operation of multiplication will be discussed in the next unit.

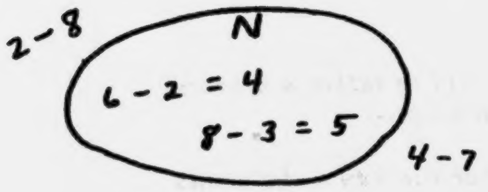
OPERATION OF SUBTRACTION

The natural number n for the missing addend that makes 5 + n = 8 a true statement is called the difference '8-5', and the operation of finding n is called subtraction. The difference '8-5' is another name for the number 3 in the set of natural numbers and therefore '8-5' may be replaced by '3'.

5 + ♦ = 8
♦ = 8 - 5
♦ = 3

The set N of natural numbers is closed under the operations of addition and multiplication. This is the same as saying that the sum or product of two numbers of the set N is a number of the set. Both addition and multiplication in the set are commutative and associative.

Are the principles which are valid for addition also valid for subtraction in the set of natural numbers?



If 8 + □ = 2, then □ = 2 - 8. The replacement for □ is not a natural number. Therefore, the set N is not closed under the operation of subtraction.

The example 8 - 3 ≠ 3 - 8 proves that subtraction is not commutative.

The example (8 - 3) - 2 ≠ 8 - (3 - 2) proves that subtraction is not associative.

However, multiplication is distributive over subtraction and the principle may be stated:

$$a \cdot (b - c) = ab - ac \qquad \text{if } b \geqslant c$$

where a, b, and c are numbers in set N.

THE SET OF INTEGERS

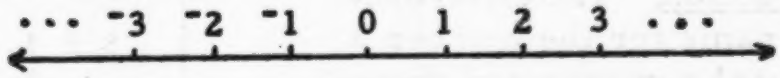
The set N = {0, 1, 2, 3, ··· } is not closed under the operation of subtraction, since the difference of two natural numbers may not exist in the set. This leads to the construction of a set of numbers that is closed under the operation of subtraction.

In most of the standard arithmetic programs, the universal set has included only the natural numbers. Children in the primary and elementary grades have been taught traditionally that a larger number cannot be subtracted from a smaller number and that in subtraction the larger number always "goes on top". However, a child's concept of number should not be so restricted.

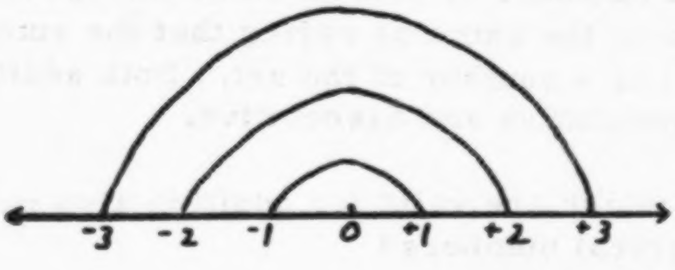
In some modern programs primary and elementary children develop intuitive notions of negative numbers through use of a number line or through other activities suggesting a need for numbers not in the set of natural numbers.

Students may be led to discover that a number line may be constructed as illustrated:

3 - 5 = □



Teachers may include activities that lead children to generalize that for each number represented to the right of zero, there is an "opposite" number represented to the left of zero and conversely.



The set of integers includes the positive integers, zero, and the negative integers. The elementary approach to the set of integers is to consider the natural numbers greater than zero as the positive integers, and the opposites to the positive integers as the negative integers. The integers are sometimes called "signed numbers".

A more formal approach is to consider the integers as a new set of numbers constructed from the set of natural numbers. An integer may then be considered as a family of differences between elements of ordered pairs of natural numbers. If (a, b) is an ordered pair of natural numbers, then a-b represents an integer.

Examples of ordered pairs of natural numbers equivalent to the integers $^+2$ or $^-2$ follow: (The symbol "$\leftrightarrow$" is read "is equivalent to")

$(3, 1) \leftrightarrow 3 - 1 \leftrightarrow {}^+2$ $\qquad$ $(1, 3) \leftrightarrow 1 - 3 \leftrightarrow {}^-2$

$(8, 6) \leftrightarrow 8 - 6 \leftrightarrow {}^+2$ $\qquad$ $(6, 8) \leftrightarrow 6 - 8 \leftrightarrow {}^-2$

$(9, 7) \leftrightarrow 9 - 7 \leftrightarrow {}^+2$ $\qquad$ $(7, 9) \leftrightarrow 7 - 9 \leftrightarrow {}^-2$

Notice that (3, 1) is equivalent to $^+2$ and that (1, 3) is equivalent to $^-2$. Thus the ordered pairs (3, 1) and (1, 3) represent different integers. In general, if (a, b) is an ordered pair of natural numbers, then

$a-b$ represents a positive integer if $a > b$,

$a-b$ represents 0 if $a = b$, and

$a-b$ represents a negative integer if $a < b$.

The set of integers may be represented:

$$\{\ldots, {}^-4, {}^-3, {}^-2, {}^-1, 0, {}^+1, {}^+2, \ldots\}$$

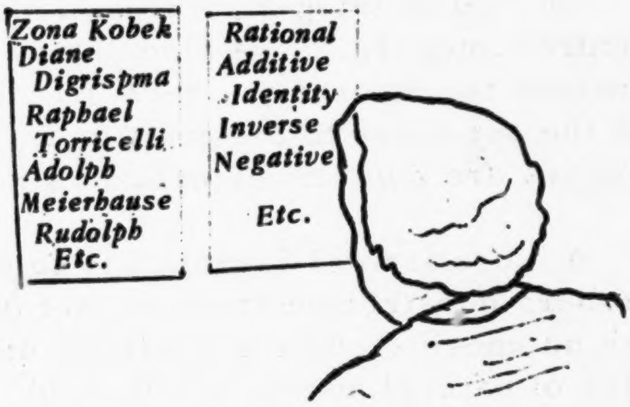
ADDITIVE INVERSES

Two integers are called additive inverses of each other if their sum is 0. For example, $^+8$ is the additive inverse of $^-8$ and $^-8$ is the additive inverse of $^+8$ since $^+8 + {}^-8 = 0$. Every integer has an additive inverse. Since $0 + 0 = 0$, 0 is its own additive inverse.

This may be stated as a new principle: For all a, where a is an integer, there exists a negative of a such that

$$a + {}^-a = 0$$

TERMINOLOGY OF MATHEMATICS



When a number, relationship, property, or principle is important enough to be distinguished from other recognized concepts, it needs to be assigned a special name. The choice of name may be arbitrary, or it may be suggested by the concept to be named.

The name "additive inverse" is used to convey the sense of an opposite in addition. Also the name serves to distinguish the inverse which has its origin in addition from other inverses in mathematics. Terminology thus suggested by relationships and concepts results in a more meaningful vocabulary in mathematics.

THE NATURAL NUMBERS AND THE INTEGERS

There is a one-to-one correspondence between the elements of the set of natural numbers greater than zero and of the set of positive integers.

$1 \sim {}^{+}1 \qquad 2 \sim {}^{+}2 \qquad 3 \sim {}^{+}3$

The principles that were stated for the set of natural numbers also hold for the set of positive integers and zero.

ADDITION OF INTEGERS

Intuitive notions of addition of integers may be developed through use of a number line or a thermometer scale.

Try to discover how to add integers by studying the examples below.

${}^{+}5 + {}^{+}3 = {}^{+}8 \qquad {}^{-}5 + {}^{+}3 = {}^{-}2$

${}^{+}5 + {}^{-}3 = {}^{+}2 \qquad {}^{-}5 + {}^{-}3 = {}^{-}8$

SUBTRACTION OF INTEGERS

The operation of subtraction on the elements of the set of integers may be defined in terms of its inverse operation addition.

Addition	Subtraction	Interpretation of subtraction
${}^{+}5 + {}^{+}3 = {}^{+}8$	${}^{+}5 = {}^{+}8 - {}^{+}3$	${}^{+}5 = {}^{+}8 + {}^{-}3$
${}^{+}5 + {}^{-}3 = {}^{+}2$	${}^{+}5 = {}^{+}2 - {}^{-}3$	${}^{+}5 = {}^{+}2 + {}^{+}3$
${}^{-}5 + {}^{+}3 = {}^{-}2$	${}^{-}5 = {}^{-}2 - {}^{+}3$	${}^{-}5 = {}^{-}2 + {}^{-}3$
${}^{-}5 + {}^{-}3 = {}^{-}8$	${}^{-}5 = {}^{-}8 - {}^{-}3$	${}^{-}5 = {}^{-}8 + {}^{+}3$

Subtracting an integer is equivalent to adding its additive inverse, When subtraction of an integer is defined in terms of addition of the additive inverse of the integer, the principles stated for the set of natural numbers under addition hold. This interpretation of subtraction as addition of the additive inverse is emphasized in algebra.

Notice that $^{+}7 - {}^{+}3 \neq {}^{+}3 - {}^{+}7$ but that $^{+}7 + {}^{-}3 = {}^{-}3 + {}^{+}7$ and that the commutative principle then holds.

Similarly, $({}^{+}7 - {}^{+}3) - {}^{+}2 \neq {}^{+}7 - ({}^{+}3 - {}^{+}2)$ but $({}^{+}7 + {}^{-}3) + {}^{-}2 = {}^{+}7 + ({}^{-}3 + {}^{-}2)$ and the associative principle then holds.

OPERATIONS WITH INTEGERS FORMALLY DEFINED (optional)

The integers may be represented as ordered pairs (a, b) and (c, d) of natural numbers and the following definitions given for the operations of addition, subtraction, and multiplication. This is one way of explaining the "rules of signs" in algebra.

ADDITION

Definition	Illustration	Explanation
(a, b) + (c, d) = (a+c, b+d)	(4, 7) + (7, 2) = (11, 9)	$^{-}3 + {}^{+}5 = {}^{+}2$
	(7, 4) + (5, 2) = (12, 6)	$^{+}3 + {}^{+}3 = {}^{+}6$
	(4, 7) + (2, 5) = (6, 12)	$^{-}3 + {}^{-}3 = {}^{-}6$
	(7, 4) + (1, 6) = (8, 10)	$^{+}3 + {}^{-}5 = {}^{-}2$

SUBTRACTION

Definition	Illustration	Explanation
(a, b)-(c, d) = (a+d, b+c)	(4, 7) - (7, 2) = (6, 14)	$^{-}3 - {}^{+}5 = {}^{-}8$
	(7, 4) - (6, 2) = (9, 10)	$^{+}3 - {}^{+}4 = {}^{-}1$
	(4, 7) - (2, 6) = (10, 9)	$^{-}3 - {}^{-}4 = {}^{+}1$
	(7, 4) - (1, 6) = (13, 5)	$^{+}3 - {}^{-}5 = {}^{+}8$

MULTIPLICATION

Definition	Illustration	Explanation
(a, b)·(c, d) = (ac+bd, ad+bc)	(2, 1) (2,1) = (5, 4)	$^{+}1 \cdot {}^{+}1 = {}^{+}1$
	(2, 1) (1, 2) = (4, 5)	$^{+}1 \cdot {}^{-}1 = {}^{-}1$
	(1, 2) (2, 1) = (4, 5)	$^{-}1 \cdot {}^{+}1 = {}^{-}1$
	(1, 2) (1, 2) = (5, 4)	$^{-}1 \cdot {}^{-}1 = {}^{+}1$

SUMMARY OF PRINCIPLES FOR INTEGERS

	Addition	Multiplication
Closure and Uniqueness	$a + b$ is a unique integer.	$a \cdot b$ is a unique integer
Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Identity Element	$a + 0 = 0 + a = a$	$1 \cdot a = a \cdot 1 = a$
Inverse Element	$a + {}^-a = {}^-a + a = 0$	
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

The following tables summarize for the set of natural numbers and the set of integers the principles of closure, commutativity, and associativity.

CLOSURE

Sets	Addition	Subtraction
Natural Numbers	yes	no
Integers	yes	yes

COMMUTATIVITY

Sets	Addition	Subtraction
Natural Numbers	yes	no
Integers	yes	no

ASSOCIATIVITY

Sets	Addition	Subtraction
Natural Numbers	yes	no
Integers	yes	no

REFERENCES FOR TEACHERS

UNIT V

Adler--The New Mathematics: pp. 46-61

Adler--Mathematics: Exploring the World of Number and Space: pp. 41-42

Banks--Elements of Mathematics: pp. 135-154

Larsen--Check and Double Check: pp. 1-15

Larsen--Number Games: pp. 1-15

Larsen--Other Ways: pp. 1-15

NCTM, 24th Yearbook--The Growth of Mathematical Ideas, K-12: pp. 22, 45-59

NCTM, 25th Yearbook--Instruction in Arithmetic: pp. 249-259

Osborn, DeVault, Boyd, Houston--Extending Mathematics Understanding: pp. 50-59

Swain--Understanding Arithmetic: pp. 58-65, 73-79, 170-182

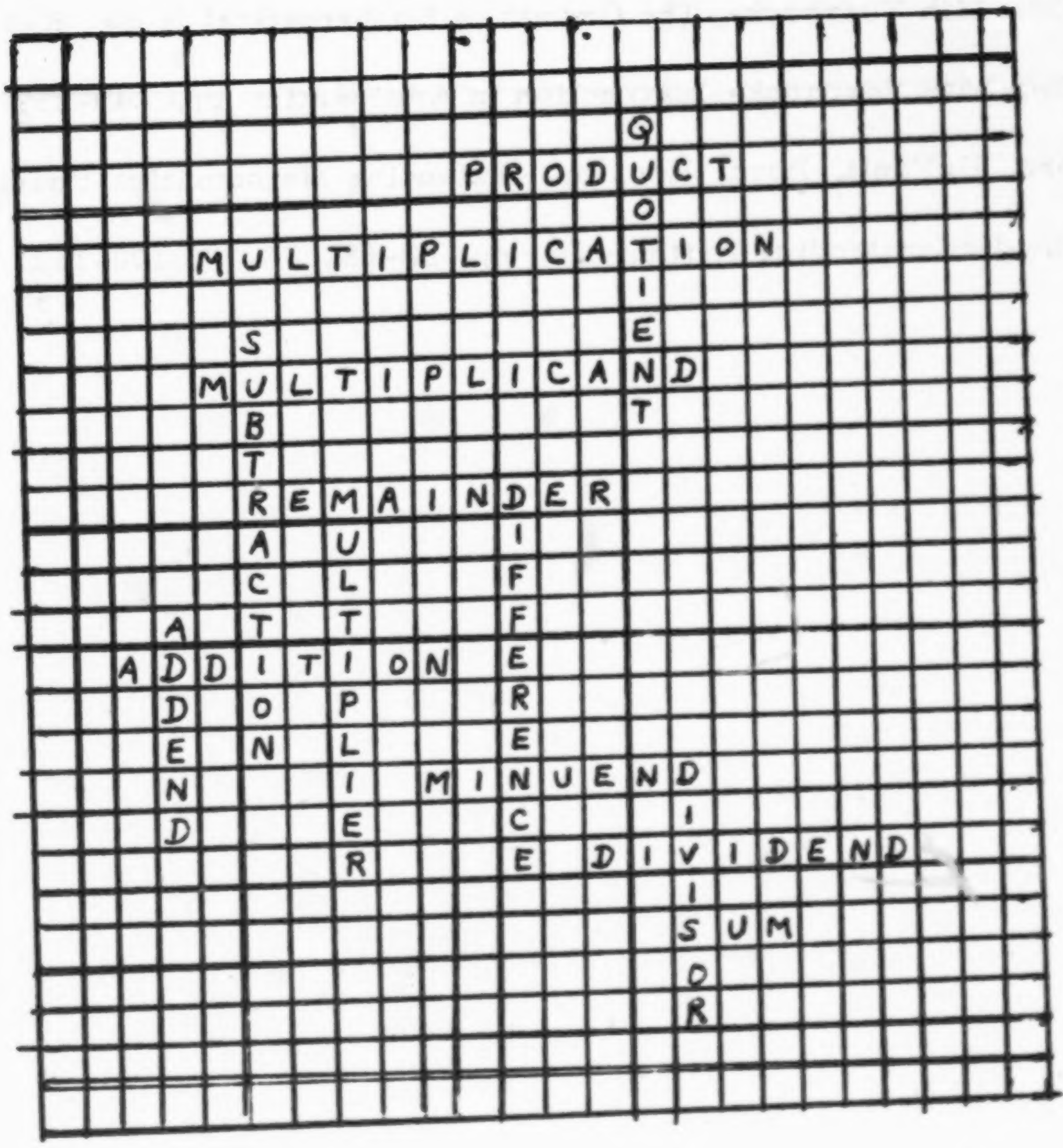
ACTIVITIES

5.01 Vocabulary Review

Give each student one sheet of 1/4 inch graph paper and tell him to build words using certain designated terms, e.g., the names of words used in addition, subtraction, multiplication, and division of numbers.

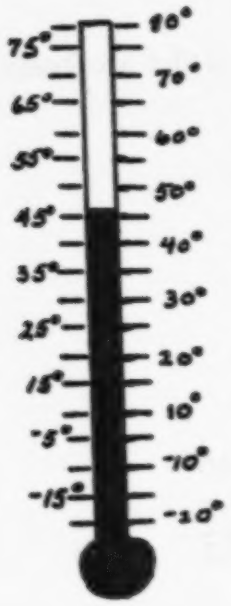
Words cannot be "run-on". If a word is misspelled, it and all words built on it are not counted. The pupil with the greatest number of correct words at the end of a given time is the winner. Words to be used may be reviewed and a list made on the board.

subtraction
multiplicand
multiplier
minuend
multiplication
remainder
product
subtrahend
difference
divisor
dividend
quotient
addend
sum


QUOTIENT
PRODUCT
MULTIPLICATION
MULTIPLICAND
SUBTRACTION
REMAINDER
MULTIPLIER
DIFFERENCE
ADDITION
ADDEND
MINUEND
DIVIDEND
DIVISOR
SUM

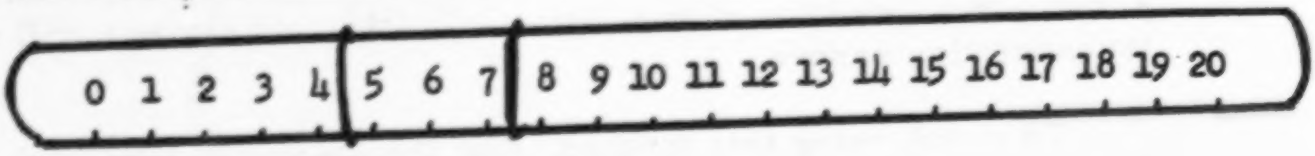
ACTIVITIES

5.02 Adjustable Thermometer



Draw a thermometer on a heavy piece of cardboard, and make slits at the top and bottom of the mercury column. Color the bulb at the bottom red. Mark off degrees at the right and left of the mercury column. Sew a piece of red and a piece of white ribbon or elastic together. Lace the ribbon through the slits and fasten the ends together at the back to make a movable loop that will slide up and down. The ribbon can then be moved so that the red part indicates the temperature.

5.03 A Primary Number Rule with Slides



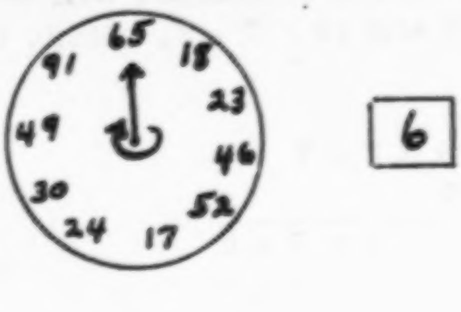
Primary number rules with slides may be constructed for primary children to use in addition and subtraction. These can easily be made from drapery rod extensions. Plastic carriers, which fit in the groove at the back, can be used for slides. Wrap wires around the rod as shown and fasten these to the plastic carriers. Press the ends of the rods so that the plastic carriers will stay in the groove. Use India ink or a magic marker to make the numerals.

Children enjoy using these to solve problems such as the following:

$6 - 3 + 5 - 7 = \square$

$8 + 2 - 6 + 7 = \square$

5.04 Clock subtraction on flannel board:



Subtract the number represented on the right from each of the numbers on the clock. Try to rotate the hand completely without missing a number

5.05 **Cryptarithms:**

Cryptarithms making use of subtraction might include the following:

a. $$\begin{array}{r} NINE \\ -\ TEN \\ \hline TWO \end{array}$$

b. $$\begin{array}{r} SEVEN \\ -\ NINE \\ \hline EIGHT \end{array}$$

Solutions for the above problems:

a. $$\begin{array}{r} 1319 \\ -\ 691 \\ \hline 628 \end{array}$$

b. $$\begin{array}{r} 21514 \\ -\ 4641 \\ \hline 16873 \end{array} \quad \text{or} \quad \begin{array}{r} 54146 \\ -\ 6764 \\ \hline 47382 \end{array}$$

5.06 **Casting Out Elevens**

The method of casting out nines fails to catch a common numerical error--transposition of two digits. Casting out elevens will detect transposition errors.

To check using this method, find the algebraic sum of the digits when alternate signs are assigned with the units digit always being assigned a positive value, the tens a negative, etc.

$$\begin{array}{rlcr} 23416 & 6 - 1 + 4 - 3 + 2 = 8 & & 8 \\ +\ 2925 & \underline{5 - 2 + 9 - 2 \quad = 10} & \text{or} & \underline{-1} \\ \hline 26341 & 1 - 4 + 3 - 6 + 2 = -4 & \text{or} & 7 \end{array} \qquad \begin{array}{l} 18 - 11 = 7 \\ -4 + 11 = 7 \end{array}$$

Actually this makes use of a divisibility test for 11 and uses the excesses of 11's in place of the original numbers and the difference.

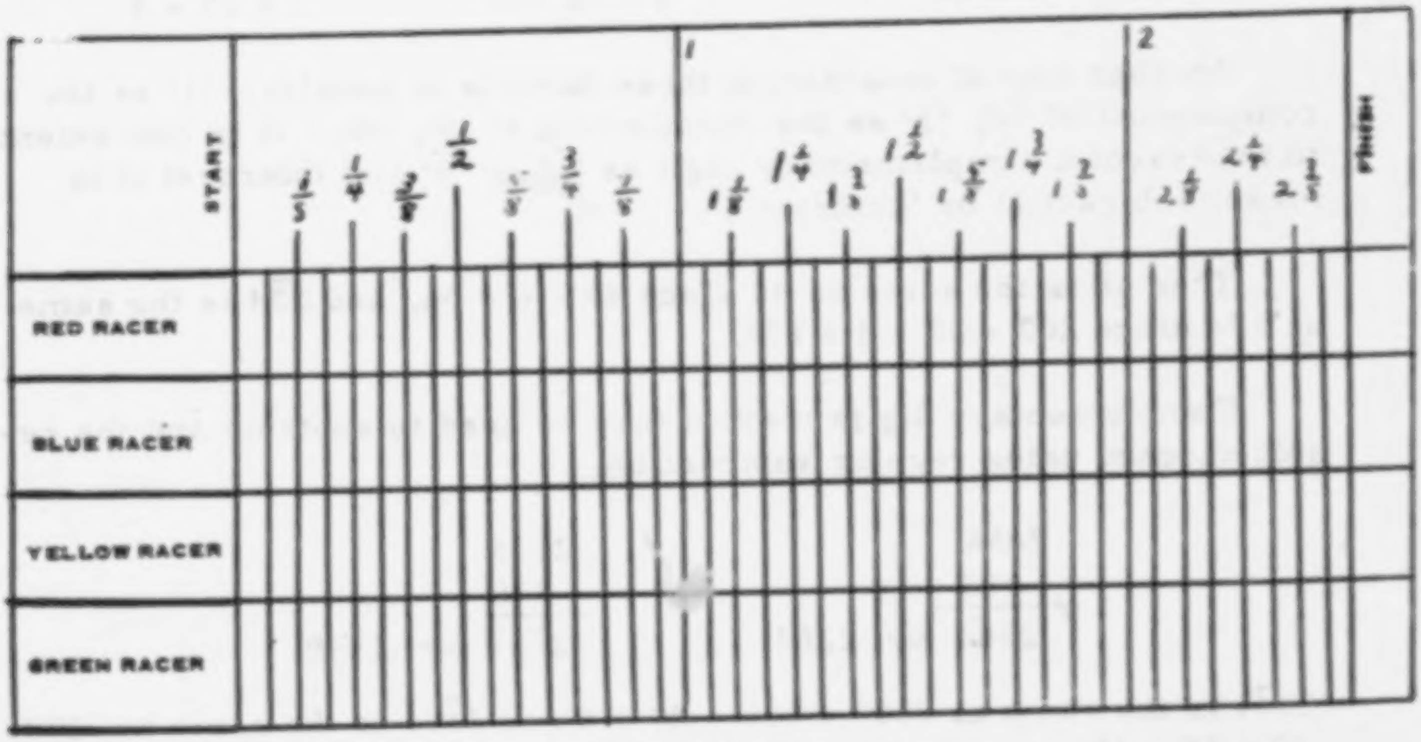
5.07 **Subtracting 9 from Teen Numbers**

A quick way to subtract 9 from any teen number is to add the digits in the teen numeral. Do you see why this method will work? A teen number is represented by 10 + n and 10 - 9 = 1. Study the examples below:

$$\begin{array}{r} 17 \\ -\ 9 \\ \hline 8 \end{array} \quad \begin{array}{r} 15 \\ -9 \\ \hline 6 \end{array} \quad \begin{array}{r} 14 \\ -\ 9 \\ \hline 5 \end{array} \qquad \begin{array}{l} 17 = 10 + 7 = 9 + 1 + 7 \\ \underline{\qquad\qquad\quad -9 \qquad\quad} \\ \qquad\qquad\qquad 1 + 7 = 8 \end{array}$$

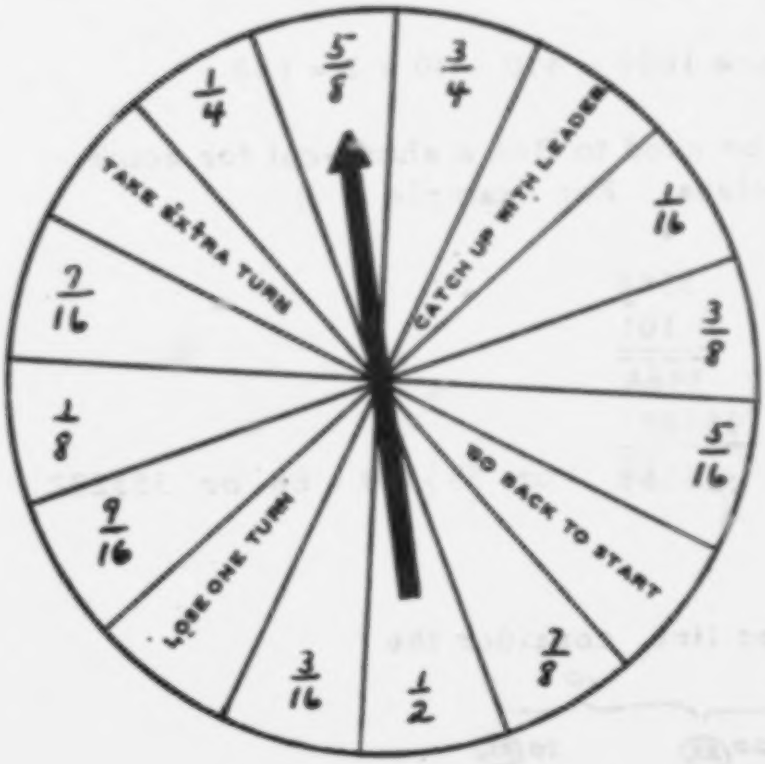
5.08 Speedway

Prepare a board and spinner similar to the diagrams Any number of students may compete, or teams may be formed. Each child or team should have a small auto, plane, missile, or marker which is placed on the starting line. Each student then takes a turn with the spinner to determine how far to advance.


START
1/8
1/4
3/8
1/2
5/8
3/4
7/8
1
1 1/8
1 1/4
1 3/8
1 1/2
1 5/8
1 3/4
1 7/8
2
2 1/8
2 1/4
2 3/8
FINISH
RED RACER
BLUE RACER
YELLOW RACER
GREEN RACER

SPEEDWAY BOARD

To review addition and subtraction students may advance to the finish line and return. Whole numbers may be used instead of fractions.


5/8
3/4
CATCH UP WITH LEADER
1/16
3/8
5/16
GO BACK TO START
7/8
1/2
3/16
LOSE ONE TURN
9/16
1/8
7/16
TAKE EXTRA TURN
1/4

SPINNER

Each pupil may keep a score sheet similar to the one below:

BLUE RACER				
SPIN	1	2	3	4
	0 + 1/8 3/8	3/8 + 1/4 5/8	5/8 LOSE TURN	5/8 7/16 1 1/16

SCORE SHEET

.09 Complementary Digits

Since subtraction is the opposite of addition, the natural numbers may be expressed:

$$\begin{array}{lll} 9 = 10 - 1 & 6 = 10 - 4 & 3 = 10 - 7 \\ 8 = 10 - 2 & 5 = 10 - 5 \quad , & 2 = 10 - 8 \\ 7 = 10 - 3 & 4 = 10 - 6 & 1 = 10 - 9 \end{array}$$

Another way of considering these facts is to consider '1' as the complement of '9', '2' as the complement of '8', etc. It is convenient to represent a complementary digit as '$\bar{1}$' or '$\bar{2}$' and interpret it to mean 'subtract 1' or 'subtract 2'.

Then $4\bar{6}$ is the same as 34 since 40 - 6 = 34, and $2\bar{3}4$ is the same as 174 since 200 - 30 + 4 = 174.

Complementary digits method may be used to subtract and the result checked using regular subtraction.

$$\begin{array}{r} 2456 \\ 192 \\ \hline 23\bar{4}4 \end{array} \text{ or } 2264 \qquad\qquad \begin{array}{r} 2185 \\ 236 \\ \hline 2\bar{1}5\bar{1} \end{array} \text{ or } 1949$$

($23\bar{4}4$ is the same as 2000 + 300 - 40 + 4 and $2\bar{1}5\bar{1}$ is the same as 2000 - 100 + 50 - 1).

By using complementary digits it is possible to avoid using digits larger than 5.

247 may be written as $25\bar{3}$ since 200 + 50 - 3 = 247 .

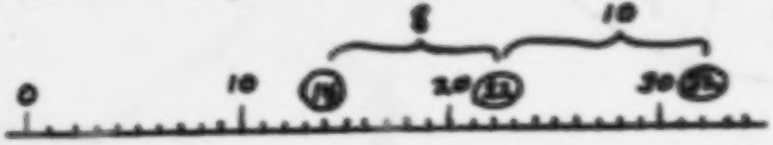
638 may be written $1\bar{4}4\bar{2}$ since 1000 - 400 + 40 - 2 = 638

Complementary digits may be used to find a short-cut for some complicated multiplication exercises. For example

$$\begin{array}{r} 3568 \\ 99 \\ \hline 32112 \\ 32112 \\ \hline 353232 \end{array} \text{ is equivalent to } \begin{array}{r} 3568 \\ 10\bar{1} \\ \hline \bar{3}\bar{5}\bar{6}\bar{8} \\ 35680 \\ \hline 3533\bar{6}\bar{8} \end{array} \text{ or } 353300 - 68 \text{ or } 353232$$

5.10 Number Line

To subtract using the number line, consider the



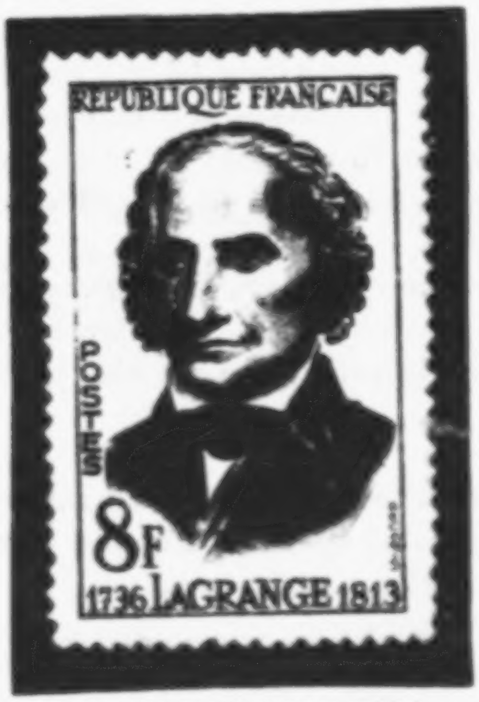
example 32 - 18; go from 32 to 22 and then 8 below 22.

5.11 Mathematicians

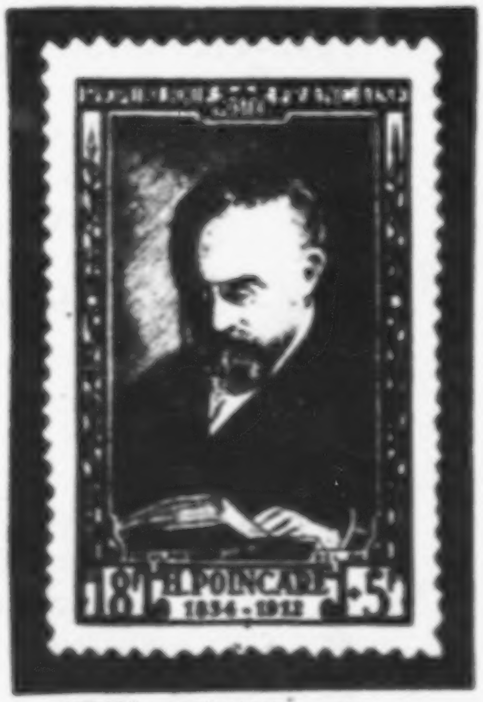
1623-1662	Pascal was a precocious genius who presented a comprehensive essay on conics at the age of 16. Find out about his famous triangle.
1642-1727	Newton, an English mathematician, considered as one of the greatest of all time, was one of the founders of the calculus. Two of his friends were Edmund Halley, who calculated the path of the comet of 1682, and Sir Christopher Wrenn, who designed St. Paul's in London.
1646-1716	Leibniz, a German mathematician, is also considered one of the founders of the calculus. He invented the binary system which is widely used on electronic computers.
1667-1754	DeMoivre, a French mathematician, was interested in probability. He discovered the bell shaped curve to which a normal distribution conforms.
1698-1746	Maclaurin, a mathematical prodigy (Scotch) who entered the University of Glasgow at the age of 11, was teaching at the college in Aberdeen at 19.
1707-1783	Euler, a Swiss mathematician, aided the Czarina of Russia in suppressing Diderot's irreligious views by using mathematics. He was a prolific writer of mathematics even after he was almost totally blind.
1711-1787	Boscovich, a Slovene, was well known both as a mathematician and as an astronomer.
1736-1813	Lagrange, a French mathematician, was invited by Frederick the Great and by Louis XVI to work in their courts. He helped work out the metric system.
1749-1827	Laplace, a French mathematician, was of poor and humble origin. He was educated by men who recognized his ability.

5.12 Stamps

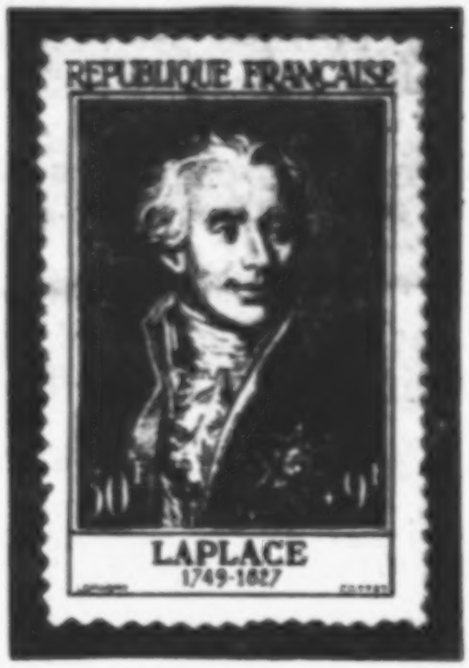
Another continuing project is the collecting of stamps. A few more are shown.


REPUBLIQUE FRANCAISE
POSTES
8F
1736 LAGRANGE 1813

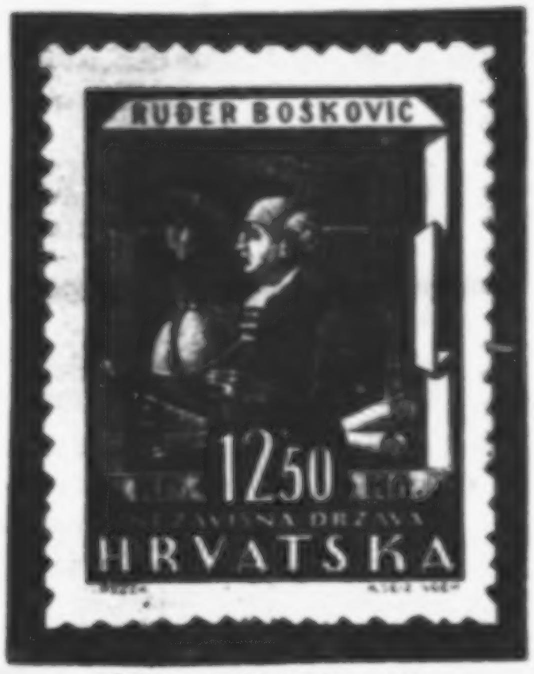
LAGRANGE HELPED WORK OUT THE METRIC SYSTEM. THE STAMP SHOWN IS AN 8-FRANC VIOLET-BLUE AND BLUE-GREEN STAMP OF 1958 ISSUED BY FRANCE.


18F
H. POINCARE
1854 - 1912
+5F

HENRI POINCARÉ WAS A FRENCH ASTRONOMER AND MATHEMATICIAN. THISSTAMP WAS ISSUED BY FRANCE ON A DARK BROWN SEMI-POSTAL ISSUE OF 1952 FOR 18 + 5 FRANCS.


REPUBLIQUE FRANCAISE
30F
+9F
LAPLACE
1749-1827

LAPLACE IS PICTURED ON THE FRENCH ROSE-BROWN 30 + 9-FRANC STAMP OF 1955. HE WROTE ABOUT SOLAR MOTIONS.


RUDER BOSKOVIC
12.50
NEZAVISNA DRZAVA
HRVATSKA

ROGER BOSCOVICH, S.J., (1711-1787) A SLOVENE, WAS WELL KNOWN AS A TALENTEDMATHEMATICIAN. STAMP WAS ISSUED BY CROATIA IN 1943 AND WAS DARK VIOLET-BROWN AND VALUED AT 12.50 KUNA.

1) Arrange in order of value:

$^+7$, $^-9$, $^+5$, $^-1$, $^-3$, 0, $^+4$, $^-2$

2) What is the additive inverse of each of the following:

(a) $^+2$ (b) $^-5$ (c) $^-6$ (d) ^+a

3) Complete the following table:

	Operation	No. Pair	Result	Explanation
a.	+	$^-6$, $^+4$	$^-2$	$(^-6) + (^+4)$
b.	-	$^+2$, $^-3$	$^+5$	$(^+2) - (^-3)$
c.	+	$^+3$, $^+7$	____	____
d.	+	$^+4$, ____	$^-1$	____
e.	-	$^+7$, $^-2$	____	____
f.	-	$^-4$, ____	$^+1$	____
g.	+	$\square$, $^-6$	$^+2$	____

4) If U = {0, 1, 2, 3, ···}, find the solution set for each of the following:

(a) $5 + \square = 8$

(b) $\square + 13 = 5$

(c) $6 - \square > 7$

(d) $\square + \triangle = 5$

5) If U = {···, $^-3$, $^-2$, $^-1$, 0, $^+1$, $^+2$, $^+3$, ···}, find the solution set for each of the following:

(a) $^+5 + \square = {}^+8$

(b) $\square + {}^+13 = {}^+5$

(c) $^+6 - \square > {}^+7$

(d) $\square + \triangle = {}^+5$

EXERCISES (answers)

1) $^{-}9$, $^{-}3$, $^{-}2$, $^{-}1$, 0, $^{+}4$, $^{+}5$, $^{+}7$

2) (a) $^{-}2$;

(b) $^{+}5$;

(c) $^{+}6$;

(d) ^{-}a

3)

c.	+	$^{+}3, ^{+}7$	$^{+}10$	$(^{+}3) + (^{+}7)$
d.	+	$^{+}4, ^{-}5$	$^{-}1$	$(^{+}4) + (^{-}5)$
e.	-	$^{+}7, ^{-}2$	$^{+}9$	$(^{+}7) - (^{-}2)$
f.	-	$^{-}4, ^{-}5$	$^{+}1$	$(^{-}4) - (^{-}5)$
g.	+	$^{+}8, ^{-}6$	$^{+}2$	$(^{+}8) + (^{-}6)$

4) (a) $\{3\}$

(b) $\{\ \}$

(c) $\{\ \}$

(d) $\{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\}$

5) (a) $\{^{+}3\}$

(b) $\{^{-}8\}$

(c) $\{^{-}2, ^{-}3, ^{-}4, ^{-}5, \cdots\}$

(d) An infinite number of solutions such as $(0, ^{+}5)$, $(^{+}1, ^{+}4)$, $(^{-}6, ^{+}11)$, etc.

ASSIGNMENT **UNIT V**

1. Arrange the following in order of value from least to greatest:

$^-4, \ ^+6, \ ^-2, \ ^+4, \ ^+1, \ ^+3, \ ^-1, \ 0$

2. What is the additive inverse of each of the following:

(a) $^+3$; (b) $^-2$; (c) $^-5$; (d) 0; (e) ^-a

3. If $U = \{\cdots, {}^-2, {}^-1, 0, {}^+1, {}^+2, \cdots\}$ find the solution set for each of the following:

a. $^+12 + n = {}^+9$

b. $^+12 + n < {}^+9$

c. $^+12 + n > {}^+9$

d. $\square + {}^-3 < 0$

e. $^-2 \leq \square < {}^+3$

f. $^+3 = \square + {}^-3$

4. Complete the following table:

	Operation	Number Pair	Result	Explanation
a.	+	$^-3, {}^-3$	$^-6$	$(^-3) + (^-3)$
b.	-	$^-6, {}^+4$	$^-10$	$(^-6) - (^+4)$
c.	+	$^-6, {}^+4$	______	______
d.	-	$^-3, {}^-3$	______	______
e.	+	$^-8, {}^+2$	______	______
f.	-	$^+4, {}^-3$	______	______
g.	+	$^+4, {}^-8$	______	______
h.	-	___, $^-5$	$^-3$	______
i.	-	______	______	$(^-6) - (^+5)$

5. It can be assumed that the set of positive integers is closed under the operation of multiplication. Thus the product of two positive integers is a positive integer. For example, $^+5\ {}^+2 = {}^+10$. Use this information and the order property of integers to complete each of the following exercises:

(a)
$^+3 \cdot {}^+1 =$ ______
$^+2 \cdot {}^+1 =$ ______
$^+1 \cdot {}^+1 =$ ______
$0 \cdot {}^+1 =$ ______
$^-1 \cdot {}^+1 =$ ______
$^-2 \cdot {}^+1 =$ ______
$^-3 \cdot +1 =$ ______

(b)
$^+1 \cdot {}^+3 =$ ______
$^+1 \cdot {}^+2 =$ ______
$^+1 \cdot {}^+1 =$ ______
$^+1 \cdot 0 =$ ______
$^+1 \cdot {}^-1 =$ ______
$^+1 \cdot {}^-2 =$ ______
$^+1 \cdot {}^-3 =$ ______

(c)
$^+3 \cdot {}^-1 =$ ______
$^+2 \cdot {}^-1 =$ ______
$^+1 \cdot {}^-1 =$ ______
$0 \cdot {}^-1 =$ ______
$^-1 \cdot {}^-1 =$ ______
$^-2 \cdot {}^-1 =$ ______
$^-3 \cdot {}^-1 =$ ______

If the order of the products in each exercise is to be preserved, how must each of the following products be defined?

(d) a negative number times a positive number ________
(e) a positive number times a negative number ________
(f) a negative number times a negative number ________

6. In a subtraction problem □ - △ = ⏢

   a. What happens to ⏢ if □ and △ are both increased by ▽ ?
   b. What happens to ⏢ if both □ and △ are doubled?
   c. What happens to ⏢ if only □ is doubled?
   d. Can you predict what △ + ⏢ will equal?
   e. Can you predict what □ - ⏢ will equal?
   f. If △ is decreased by ▽ and the result subtracted from □ , what would the remainder be?

7. Use regrouping to perform the operation of subtraction:

   a. 4682_{10}
   946_{10}

   d. 364_7
   146_7

   b. Check by casting out 9's.
   c. Check by casting out 11's

   e. Check by casting out 6's.
   f. Check by casting out 11's in base 7. ($11_{\text{base } 7} = 8_{\text{base } 10}$

8. Express each of the following as a numeral in the indicated base

   a. $146_8 = \_\_\_\_\_\_\_\__{10}$
   b. $82_{10} = \_\_\_\_\_\_\_\__3$
   c. $11011_2 = \_\_\_\_\_\_\_\__{10}$
   d. $241_{10} = \_\_\_\_\_\_\_\__6$

9. Let A = the set of girl scouts = {Mary, Helen, Betty, Carol}
   Let B = the set of all 8th grade girls = {Jill, Carol, Joan, Phyllis}
   Let C = the set of all girls taking band = {Mary, Jill, Carol, Sue}

Draw Venn diagrams and write names in regions. Then answer the following questions.

   a. Which girl is in the 8th grade and is taking band and belongs to the Girl Scouts?
   b. Which girls are not taking band?
   c. Which 8th grade girls are not in band or scouting?
   d. Is there an 8th grade girl in scouting who is not in band?

10. Complete the following "magic" squares:

   a. Use numerals representing numbers from 22 - 37.

	23	24	
30			

   b. Use numerals representing numbers from 32 - 56.

		32		

(1) $^{-}4$, $^{-}2$, $^{-}1$, 0, $^{+}1$, $^{+}3$, $^{+}4$, $^{+}6$

(2) (a) $^{-}3$; (b) $^{+}2$; (c) $^{+}5$; (d) 0; (e) ^{+}a

(3) a. $\{^{-}3\}$
b. $\{^{-}4, ^{-}5, ^{-}6, \cdots\}$ or the set of all integers less than $^{-}3$.
c. $\{^{-}2, ^{-}1, 0, ^{+}1, ^{+}2, \cdots\}$ or the set of all integers greater than $^{-}3$.
d. $\{^{+}2, ^{+}1, 0, ^{-}1, ^{-}2, \cdots\}$ or the set of all integers less than $^{+}3$.
e. $\{^{-}2, ^{-}1, 0, ^{+}1, ^{+}2\}$
f. $\{^{+}6\}$

(4) c.	+	$^{-}6$, $^{+}4$	$^{-}2$	$(^{-}6) + (^{+}4)$
d.	-	$^{-}3$, $^{-}3$	0	$(^{-}3) - (^{-}3)$
e.	+	$^{-}8$, $^{+}2$	$^{-}6$	$(^{-}8) + (^{+}2)$
f.	-	$^{+}4$, $^{-}3$	$^{+}7$	$(^{+}4) - (^{-}3)$
g.	+	$^{+}4$, $^{-}8$	$^{-}4$	$(^{+}4) + (^{-}8)$
h.	-	$^{-}8$, $^{-}5$	$^{-}3$	$(^{-}8) - (^{-}5)$
i.	-	$^{-}6$, $^{+}5$	$^{-}11$	$(^{-}6) - (^{+}5)$

(5) (a)	(b)	(c)	
$^{+}3$	$^{+}3$	$^{-}3$	(d) negative
$^{+}2$	$^{+}2$	$^{-}2$	(e) negative
$^{+}1$	$^{+}1$	$^{-}1$	(f) positive
0	0	0	
$^{-}1$	$^{-}1$	$^{+}1$	
$^{-}2$	$^{-}2$	$^{+}2$	
$^{-}3$	$^{-}3$	$^{+}3$	

(6) a. Remains the same.
b. Doubled.
c. Increased by ▭
d. □
e. △
f. ⏢ + ▽

(7) a.

$$\begin{array}{r} 4682 \\ -946 \\ \hline 3736 \end{array}$$

b. 9's
4 + 6 + 8 + 2 —— 2
9 + 4 + 6 —— 1
3 + 7 + 3 + 6 —— 1

c. 11's
2 - 8 + 6 - 4 —— $^{-}4$ —— 7
6 - 4 + 9 —— 11 —— 0
6 - 3 + 7 - 3 —— 7

d.

$$\begin{array}{r} 364_7 \\ -146_7 \\ \hline 215_7 \end{array}$$

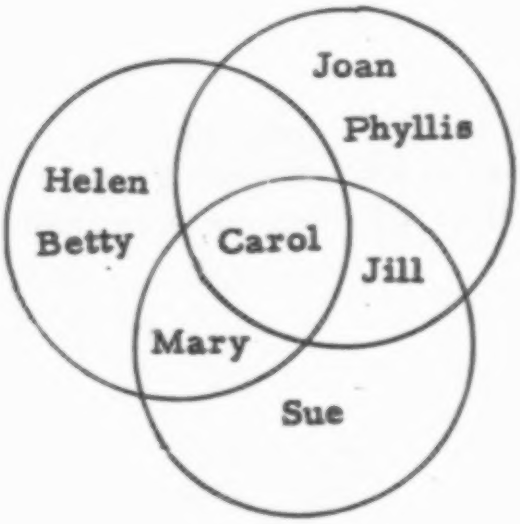
e. 6's seven
3 + 6 + 4 —— 1 —— 10_7
1 + 4 + 6 —— 5_7
2 + 1 + 5 —— 2_7

f. 11's seven
4 - 6 + 3 —— 1 —— 12_7
6 - 4 + 1 —— 3_7
5 - 1 + 2 —— 6_7

ASSIGNMENT **(answers)**

8. a. 102_{10} c. 27_{10}
   b. 10001_3 d. 1041_6

9.


Joan
Phyllis
Helen
Betty
Carol
Jill
Mary
Sue

a. Carol
b. Helen, Betty, Joan, Phyllis
c. Joan, Phyllis
d. No.

10. a.

37	23	24	34
26	32	31	29
30	28	27	33
25	35	36	22

48	55	32	39	46
54	36	38	45	47
35	37	44	51	53
41	43	50	52	34
42	49	56	33	40

UNIT VI

OPERATION OF DIVISION

UNIT VI. OPERATION OF DIVISION

> *"One for you, and one for me,*
> *and one for sister Annie."*

INTRODUCTION

In this unit the set of rational numbers, a set constructed from the integers, will be introduced to obtain closure under the operation of division.

DIVISION OF NATURAL NUMBERS

In some of the modern programs division of natural numbers is being introduced as a series of repeated subtractions to help children better understand the usual division algorithm. (Modern desk calculators operate on this principle.)

To find the solution set for the open sentence 2508 ÷ 33 = ☐ , the child may proceed in this manner.

```
33 |2508 |
    2310 |  70
    ----
     198 |
     165 |   5
    ----
      33 |
      33 |   1
  ------------
            76
```

```
       76
      ---
        1
        5
       70
33 |2508
    2310
    ----
     198
     165
    ----
      33
      33
    ----
```

He makes an "intelligent guess" of how many 33's are in 2500, e.g., 70. Then he may write '70' at the top or side as shown. He multiplies 33 by 70 and subtracts this product from 2508. Since the remainder is larger than 33, he makes a "guess" of 5 as the number of 33's in 198. He writes the '5' as shown. He multiplies 33 by 5 and subtracts this from 198. The remainder is 33 and then he writes '1'. The quotient 76 equals 70 plus 5 plus 1.

To solve the same problem, another child may "guess" differently as shown at the right, but the quotients are the same.

```
33 |2508 |
    1650 |  50
    ----
     858 |
     660 |  20
    ----
     198 |
     198 |   6
   -----------
            76
```

```
      76
     ---
       6
      20
      50
33|2508
   1650
   ----
    858
    660
   ----
    198
    198
   ----
```

2508 ÷ 33 = 76

Naturally as a child progresses, his "guesses" are expected to improve and he should look for ways of perfecting his "guessing." Many think that this method of introducing division is more meaningful because it actually makes clear that 33 is multiplied by 7 tens and 6 ones.

$$\begin{array}{r|r|r} 33 & 2508 & 70 \\ & \underline{2310} & \\ & 198 & 6 \\ & 198 & \\ \hline & & 76 \end{array}$$

Regardless of whether this method or the traditional one is used, the teacher should use correct terminology. The "goes into" description of division not only conveys the wrong mathematical idea but also degenerates too easily into "guzinto".

OPERATION OF DIVISION

Division is the inverse operation of multiplication, and multiplication is the inverse operation of division. If $10 \div 5 = 2$, then $2 \times 5 = 10$.

If a number can be found that makes the sentence $3 \cdot n = 18$ a true statement, this number is called the "quotient of 18 divided by 3" and written as '$18 \div 3$' or '18/3'. This quotient may be replaced by its standard name '6'.

If $3 \cdot n = 18$, the operation of finding n is called division. Finding the missing factor in the expression $3 \cdot n = 18$ emphasizes division as the inverse operation of multiplication.

Do the special properties which hold for multiplication also hold for division?

Is the set of integers closed under the operation of division? Is there always an integer which expresses the quotient of two integers?

While '$18 \div 3$' is another name for the integer 6, '$18 \div 5$' and '$4 \div 7$' are not names of integers. This proves that the set of integers is not closed under the operation of division.

Is the operation of division on elements of the set of integers commutative?

The example $8 \div 4 \neq 4 \div 8$ proves that the operation of division is not commutative.

Is the operation of division on elements of the set of integers associative?

The example $(16 \div 4) \div 2 \neq 16 \div (4 \div 2)$ proves that division is not associative since $4 \div 2 \neq 16 \div 2$.

Is the operation of division distributive from the left over the operations of addition and subtraction?

$24 \div (6 + 2) \neq (24 \div 6) + (24 \div 2)$ since $24 \div 8 \neq 4 + 12$
$24 \div (6 - 4) \neq (24 \div 6) - (24 \div 4)$ since $24 \div 2 \neq 4 - 6$

This illustrates that the operation of division is not distributive from the left over the operations of addition and subtraction.

However, the operation of division is distributive from the right over the operations of addition and subtraction.

$$(30 + 6) \div 2 = (30 \div 2) + (6 \div 2) = 18$$
$$(30 - 6) \div 2 = (30 \div 2) - (6 \div 2) = 12$$

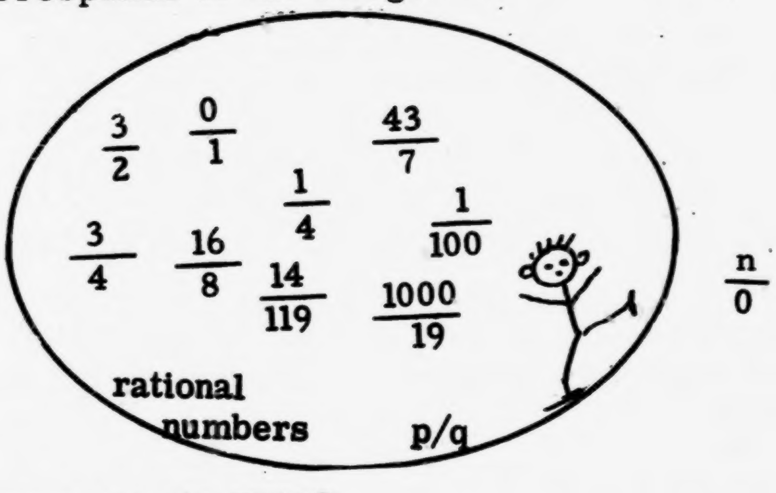
THE SET OF RATIONAL NUMBERS

It was shown in the last unit how the set of integers was constructed from the set of natural numbers.

Examine the open sentence $2n = 1$. There is no integer that can be substituted for n in $2n = 1$ to make a true statement.

This shows that the set of integers is not closed under the operation of division. In order that the operation of division (except division by zero) will always be possible, another set of numbers, the set of rational numbers, may be constructed from the integers.

A rational number will be defined as any number that may be represented as a/b where a and b are integers with $b \neq 0$. The rational number a/b ($b \neq 0$) is a fractional number. The rational number a/1 corresponds to the integer a.



The rational numbers are closed under the operations of addition, subtraction, multiplication, and division. Division by 0 is excluded.

DIVISION BY ZERO

Division by zero is excluded in our number system. The quotient $n/0 = q$ indicates that a unique whole number q must exist such that $q \times 0 = n$. There can be no unique number q because $q \times 0$ always equals 0. For example, $6/0 \neq 6$ since $0 \times 6 \neq 6$ and $6/0 \neq 0$ since $0 \times 0 \neq 6$.

Although it may appear that $0/0 = 0$ because $0 \times 0 = 0$, it also follows that 0/0 would not be a unique quotient but that it could be 4 or 6 or 7 or any number.

Clearly, division by 0 cannot be defined so that it leads to a unique result. Therefore, division by 0 is excluded. This means that 0 may never appear as the denominator of a fraction.

The following is a summary of operations with 0:

1. $0 + a = a + 0 = a$
2. $0 \cdot a = a \cdot 0 = 0$
3. $0 \div a = 0 \ (a \neq 0)$
4. $a \div 0$ Not defined
5. $0 \div 0$ Not defined
6. 0^0 Not defined

MULTIPLICATIVE INVERSE

For any integer a, except 0, the value of n in the open sentence $a \cdot n = 1$ is expressed by '1/a' and may be read "1 divided by a". Observe that $a \cdot (1/a) = 1$. The number expressed by '1/a' is called the multiplicative inverse of a. This may also be called the reciprocal of a.

The multiplicative inverse of 3 is 1/3 and the multiplicative inverse of 1/3 is 1 ÷ 1/3 or 3. Note that 1/3 x 3 = 1 and that 3 x 1/3 = 1. Likewise, the multiplicative inverse of 8/7 is 1 ÷ 8/7 or 7/8. Note that 8/7 x 7/8 = 1 and that 7/8 x 8/7 = 1.

With the extension to the rational numbers, a new principle may be stated. For all a ($a \neq 0$) where a is a rational number there exists another rational number called the multiplicative inverse of a such that

$$a \cdot \frac{1}{a} = 1$$

FRACTIONAL NUMBERS AND FRACTION NUMERALS

A fractional number is the idea which is represented by a fraction numeral or fraction. However, when no confusion results fractional numbers are often referred to as fractions.

The replacement for n that makes $a \cdot n = b$ a true statement is b/a. The 'b' is called the numerator and the 'a' the denominator.

Like fractions refer to symbols that have as denominators the same numeral. Likeness is not a property of numbers but a property of numerals. For example, '3/8' and '5/8' are like fractions, while '3/8' and '5/9' are unlike fractions. The unlike numerals '3/4' and '6/8' represent the same number.

The multiplicative identity one is used to change two or more unlike fractions to like fractions. The use of the multiplicative identity is sometimes referred to as the "Law of One". This principle is used to change '3/4' to '6/8'.

$$\frac{3}{4} \times 1 = \frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$$

This application of the multiplicative identity is often called the Fundamental Principle of Fractions: A fraction a/b is not changed in value when the numerator and denominator are multiplied by the same number k, with $k \neq 0$.

$$\frac{a}{b} = \frac{ak}{bk}\ (k \neq 0 \text{ and } b \neq 0) \text{ because } \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{k}{k} = \frac{ak}{bk}$$

FINDING A LEAST COMMON DENOMINATOR

If two fractional numbers are to be added or subtracted, they are usually represented first by like fraction numerals. Finding the least common denominator is a problem of finding the least common multiple (L. C. M.) of the numbers represented by the denominators. The least common multiple is the smallest number that is divisible by all the numbers being considered.

Three methods of finding the least common multiple are illustrated:

Method 1: Finding the L. C. M. by using prime factors

To find the L. C. M. of 14, 12, and 45, separate each number into its prime factors. Then the least common multiple will be the product of the different prime factors with each factor used the greatest number of times it occurs as a factor of any one number. In this illustration the L. C. M. will be 2 x 2 x 3 x 3 x 5 x 7 or 1260.

14 = 2 x 7

12 = 2 x 2 x 3

45 = 3 x 3 x 5

This method of finding the least common denominator is of particular importance since it is the one ordinarily used in algebra.

Method 2: Finding the L. C. M. by using intersection of sets

To find the L. C. M. of 12 and 8, for example, represent a set of multiples of 12 and a set of multiples of 8. Then find the intersection of the two sets.

The least number represented in the set resulting from the intersection of the sets will be the least common multiple.

$$A = \{12, 24, 36, 48, \cdots\}$$

$$B = \{8, 16, 24, 32, 40, 48, \cdots\}$$

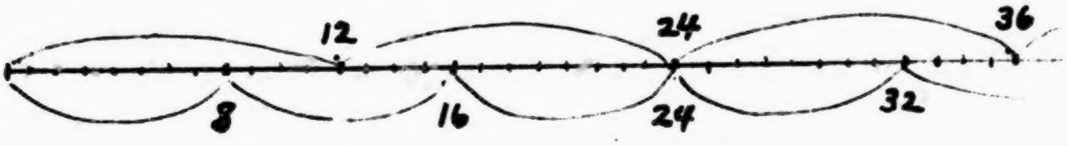
$$A \cap B = \{24, 48, \cdots\}$$

The L. C. M. is 24.

Method 3: Finding the L. C. M. by using a number line

The L. C. M. of 12 and 8 may be found on a number line by showing

Multiples of 12 and 8 may be indicated as shown below. The L. C. M. is 24 since 24 corresponds to the first point where the paths intersect.



ADDITION AND SUBTRACTION OF FRACTIONAL NUMBERS

A readiness for both addition and subtraction of fractions may be developed in the primary grades through activities on a number line or through activities based upon other visual aids. In the upper grades students may be led to discover the mathematical principles underlying these operations with fractions. The use of the multiplicative identity, the multiplicative inverse, and the distributive law will be applied to the example 3/4 + 5/6.

In the addition or subtraction of fractions the multiplicative identity is used to changed "unlike fractions" to "like fractions".

$$\begin{array}{lcl} \frac{3}{4} = \frac{?}{12} & \qquad & \frac{3}{4} \times \frac{3}{3} = \frac{9}{12} \\[1em] \frac{5}{6} = \frac{?}{12} & & \frac{5}{6} \times \frac{2}{2} = \frac{10}{12} \\ \hline & & \qquad\quad \frac{19}{12} \end{array}$$

Before applying the distributive principle to the addition of 9/12 and 10/12, interpret 9/12 as 9 x 1/12 and 10/12 as 10 x 1/12 since 1/12 is the multiplicative inverse of 12.

$$\frac{9}{12} + \frac{10}{12} = 9 \cdot \frac{1}{12} + 10 \cdot \frac{1}{12} = (9 + 10) \cdot \frac{1}{12} = 19 \cdot \frac{1}{12} = \frac{19}{12}$$

The same principles apply when sets are used in the addition of unlike fractions. To add 3/4 and 5/6 using sets, first indicate a set of numerals representing 3/4 and one representing 5/6. Then choose from each set fraction numerals with like denominators. A third set may be indicated to represent the sum 19/12.

$$\frac{3}{4}: \{\frac{'3'}{4}, \quad \frac{'6'}{8}, \quad \frac{'9'}{12}, \quad \frac{'12'}{16}, \quad \frac{'15'}{20}, \quad \frac{'18'}{24}, \ldots\}$$

$$\frac{5}{6}: \{\frac{'5'}{6}, \quad \frac{'10'}{12}, \quad \frac{'15'}{18}, \quad \frac{'20'}{24}, \quad \frac{'25'}{30}, \ldots\}$$

$$\frac{19}{12}: \{\frac{'19'}{12}, \quad \frac{'38'}{24}, \ldots\}$$

The methods used in the addition of fractional numbers also apply to the subtraction of fractional numbers. So called "borrowing" in the subtraction of fractional numbers, which is a problem of regrouping, is illustrated in solving 3 2/3 - 1 3/4.

$$
\begin{array}{rcl}
3\frac{2}{3} = 3 + \frac{8}{12} = (2+1) + \frac{8}{12} = 2 + \frac{12}{12} + \frac{8}{12} & = & 2\frac{20}{12} \\
1\frac{3}{4} = 1 + \frac{9}{12} & = & 1\frac{9}{12} \\
\hline
 & & 1\frac{11}{12}
\end{array}
$$

MULTIPLICATION AND DIVISION OF FRACTIONAL NUMBERS

A readiness for multiplication and division of fractional numbers may be developed through activities using a number line or other visual aids such as pictures of rectangles divided into squares.

Later, multiplication of fractions may be defined:

$$\frac{\text{numerator x numerator}}{\text{denominator x denominator}}$$

For example, $\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$

In some cases the computation may be simplified by factoring and applying the Law of One as shown in this illustration:

$$\frac{3}{4} \times \frac{2}{3} = \frac{3 \times 2}{4 \times 3} = \frac{3 \times 2 \times 1}{2 \times 2 \times 3} = \frac{3}{3} \times \frac{2}{2} \times \frac{1}{2} = 1 \times 1 \times \frac{1}{2} = \frac{1}{2}$$

For many students division of fractions seems to be more difficult than multiplication of fractions. Division of fractions may be taught meaningfully if students are given exercises that will allow them to discover the "rule" to invert the divisor and multiply when dividing by a fraction.

Three solutions for 3/4 ÷ 2/3 using the Law of One are given:

$\frac{3}{4} \div \frac{2}{3}$ may be written as $\dfrac{\frac{3}{4}}{\frac{2}{3}}$

Solution 1

$$\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{\frac{3}{4} \times 1}{\frac{2}{3} \times 1}$$

$$= \frac{\frac{3}{4} \times \frac{3}{3}}{\frac{2}{3} \times \frac{4}{4}}$$

$$= \frac{\frac{9}{12}}{\frac{8}{12}} \times 1$$

$$= \frac{\frac{9}{12}}{\frac{8}{12}} \times \frac{12}{12}$$

$$= \frac{9}{8}$$

$$= 1\frac{1}{8}$$

Solution 2

$$\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{\frac{3}{4}}{\frac{2}{3}} \times 1$$

$$= \frac{\frac{3}{4}}{\frac{2}{3}} \times \frac{12}{12}$$

$$= \frac{9}{8}$$

$$= 1\frac{1}{8}$$

Solution 3

$$\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{\frac{3}{4}}{\frac{2}{3}} \times 1$$

$$= \frac{\frac{3}{4}}{\frac{2}{3}} \times \frac{\frac{3}{2}}{\frac{3}{2}}$$

$$= \frac{\frac{3}{4} \times \frac{3}{2}}{1}$$

$$= \frac{3}{4} \times \frac{3}{2}$$

$$= \frac{9}{8}$$

$$= 1\frac{1}{8}$$

The last solution shows why the familiar rule to invert the divisor and multiply gives the correct result in the division of fractions.

Can you explain why like fractions are used in the division of fractional numbers? The algorithm for this method of division follows:

$$\frac{\text{numerator} \div \text{numerator}}{\text{denominator} \div \text{denominator}}$$

Then $3/4 \div 2/3$ may be written as $9/12 \div 8/12$

and $\frac{9 \div 8}{12 \div 12} = \frac{9 \div 8}{1} = 1\frac{1}{8}$

In modernized algebra courses students prove the theorem:

$$a \div b = a \times 1/b \quad (b \neq 0)$$

This states that to divide by a number is to multiply by its multiplicative inverse (reciprocal).

Then $\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$

In arithmetic students may sometimes wish to use the inverse operation and the multiplicative inverse as in these examples:

$$6 \div \frac{1}{3} = 6 \times 3$$

$$6 \times \frac{1}{3} = 6 \div 3$$

When division of a rational number is defined in terms of multiplication by the multiplicative inverse, the principles stated for the set of integers under multiplication hold.

Notice that $6 \div 5 \neq 5 \div 6$ but that $6 \times \frac{1}{5} = \frac{1}{5} \times 6$ and the commutative principle holds.

Likewise $(12 \div 4) \div 2 \neq 12 \div (4 \div 2)$ but $(12 \times \frac{1}{4}) \times \frac{1}{2} = 12 \times (\frac{1}{4} \times \frac{1}{2})$ and the associative principle holds.

Finally $\frac{(6 + 4)}{2} = \frac{1}{2}(6 + 4) = (6 + 4)\frac{1}{2}$ and the distributive principle holds.

ORDER OF OPERATIONS

There must be agreement regarding the order of operations if the results for examples such as $6 + 6 \div 6 = \square$ are to be unique.

Unless otherwise indicated by the use of parentheses, the operations of multiplication and division are performed first and then the operations of addition and subtraction. Thus $6 + 6 \div 6 = 7$. Can you explain why $3 + 6 \times 3 - 1 = 20$?

The operations of successive multiplications and divisions are performed in the order in which they occur. To avoid misinterpretation parentheses are often used to indicate operations that should be performed first.

The order of operations is not likely to be misinterpreted if $3 + 6 \times 3 - 1$ is written as $3 + (6 \times 3) - 1$.

RATIONAL NUMBERS AS ORDERED PAIRS OF INTEGERS (Optional)

The rational numbers may be represented as ordered pairs of integers (a, b) and (c, d) with $b \neq 0$ and $d \neq 0$. The ordered pair (a, b) corresponds to a/b and the ordered pair (c, d) to c/d. The ordered pair $(0, 1)$ corresponding to the integer 0 is the additive identity and the ordered pair $(1, 1)$ corresponding to the integer 1 is the multiplicative identity.

The operations of addition, subtraction, multiplication, and division may be defined as follows:

Definition	Numerical Example
$(a, b) + (c, d) = (ad + bc,\ bd)$	$(1, 2) + (2, 5) = (9, 10)$
$(a, b) - (c, d) = (ad - bc,\ bd)$	$(1, 2) - (2, 5) = (1, 10)$
$(a, b) \times (c, d) = (ac,\ bd)$	$(1, 2) \times (2, 5) = (2, 10)$
$(a, b) \div (c, d) = (ad,\ bc)$	$(1, 2) \div (2, 5) = (5, 4)$

These definitions hold for operations with the integers which may be represented as ordered pairs (a, 1) and (c, 1).

Definition	Numerical Example	Usual Form
$(a, 1) + (c, 1) = (a + c, 1)$	$(4, 1) + (2, 1) = (6, 1)$	$4 + 2 = 6$
$(a, 1) - (c, 1) = (a - c, 1)$	$(4, 1) - (2, 1) = (2, 1)$	$4 - 2 = 2$
$(a, 1) \times (c, 1) = (ac, 1)$	$(4, 1) \times (2, 1) = (8, 1)$	$4 \times 2 = 8$
$(a, 1) \div (c, 1) = (a, c)$	$(4, 1) \div (2, 1) = (2, 1)$	$4 \div 2 = 2$

Can you show that (0, 1) is the additive identity and that (1, 1) is the multiplicative identity?

SUMMARY OF PRINCIPLES FOR THE SET OF RATIONAL NUMBERS

For all numbers a, b, c, ••• in the set of rational numbers

	Addition	Multiplication
Closure and Uniqueness	$a + b = c$	$a \cdot b = c$
Commutative Principle	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative Principle	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Identity Elements	$a + 0 = 0 + a = a$	$1 \cdot a = a \cdot 1 = a$
Inverse Elements	$a + {}^{-}a = 0$	$a \cdot \frac{1}{a} = 1 \ (a \neq 0)$
Distributive Principle	$a(b + c) = a \cdot b + a \cdot c$	

SUMMARIZATION OF PRINCIPLES FOR THE FOUR OPERATIONS

The following tables summarize for the set of natural numbers, the set of integers, and the set of rational numbers the principles of closure, commutativity, and associativity.

CLOSURE

Sets	+	−	x	÷
Natural No.	yes	no	yes	no
Integers	yes	yes	yes	no
Rational No.	yes	yes	yes	yes if ÷ by 0 is excluded

COMMUTATIVITY

Sets	+	−	x	÷
Natural No.	yes	no	yes	no
Integers	yes	no	yes	no
Rational No.	yes	no	yes	no

ASSOCIATIVITY

Sets	+	−	x	÷
Natural No.	yes	no	yes	no
Integers	yes	no	yes	no
Rational No.	yes	no	yes	no

REFERENCES FOR TEACHERS

UNIT VI

Adler--The New Mathematics: pp. 79-89

Banks--Elements of Mathematics: pp. 154-165

Larsen--Check and Double Check, Grade 7
Queries and Queeries, Grade 8
Guzintas, Grade 8
Excursions in Arithmetic, Grade 6

NCTM, 24th Yearbook--Growth of Mathematical Ideas, K-12, pp. 29-38, 42-45

NCTM, 25th Yearbook--Instruction in Arithmetic: pp. 162-177

Osborn--Extending Mathematics Understanding: pp. 50-59, 60-86, 87-96

Smith & Ginsburg--Numbers and Numerals: pp. 35-38

Swain--Understanding Arithmetic: pp. 135-154

6.01 The 4's Game

By using four 4's represent the numbers from 1-20. Use only the natural numbers and the four operations to represent the numbers 1-10. Decimals but no exponents may be used in representing the numbers 11-20.

$1 = 44/44$ or $4/4 \times 4/4$ or $4/4 + 4 - 4$

$2 = 4/4 + 4/4$ or $(4 \times 4)/4+4)$

$3 = (4 + 4 + 4)/4$

$4 = 4 + 4 \times (4-4)$

$5 = (4 + 4 \times 4)/4$

$6 = 4 + (4 + 4)/4$

$7 = 4 + 4 - 4/4$ or $44/4 - 4$

$8 = 4 + 4 + 4 - 4$ or $(4 + 4)/(4 \div 4)$

$9 = 4 + 4 + 4/4$

$10 = (44-4)/4$

$11 = 4/.4 + 4/4$

$12 = (44 + 4)/4$

$13 = (4 - .4)/.4 + 4$

$14 = (4 + 4 \times .4)/.4$

$15 = 4 \times 4 - 4/4$ or $44/4 + 4$

$16 = 4 + 4 + 4 + 4$ or $(4 \times 4 \times 4)/4$

$17 = 4 \times 4 + 4/4$

$18 = 4/.4 + 4 + 4$

$19 = (4 + 4 - .4)/.4$

$20 = 4/.4 + 4/.4$

The numbers from 1 - 100 may be represented in this way. A teacher can create interest by writing the numerals 1 -100 on a large chart as shown below so that the children may write their answers and names on the chart beside each one.

No.	4 4's	Name
1	44 ÷ 44	Helen Jones
2		
3	(4 + 4 + 4) ÷ 4	Jimmy Miller
4	4 + 4 x (4 - 4)	Betty Lynch

6.02 Chinese Remainder Game

This game is over 2000 years old and was first found in the writings of the ancient Chinese mathematician, Sun-Tse.

Different remainder games can be constructed with the rules for the 10-11 game suggested.

Each student will need pencil and paper. The following directions should be given:

"Think of a number between 1 and 110 and write it down. Divide the number first by 10 and write the remainder, which will be called the 10-remainder. Next divide the number by 11 and again write the remainder which this time will be called the 11-remainder. By telling me your 10 and your 11-remainders in order, I will tell the original number you wrote on your paper. Do not tell me the quotient. Just tell me the remainders".

To tell the student his original number, multiply the 10-remainder by 11; add to this the 11-remainder multiplied by 100; divide this sum by 110 and the remainder will be the original number.

Example:

Student's part:
Chosen number: 57
10-remainder: 7
11-remainder: 2

Teacher's part:
$7 \times 11 = 77$
$2 \times 100 = 200$
sum 277

$110 \overline{)277}$ 2
220
57 Remainder

If the 11-remainder is 0, the number chosen will merely be 11 times the 10 remainder since it will be less than 110 and will represent the remainder.

A shortcut method may be used which students will probably discover. Merely write the 10-remainder twice and subtract from the first part the 11-remainder. The shortcut for the example follows:

Repeat 7 twice. 77
Subtract 2 from 1st 7 2
Remainder 57

This will hold if the first remainder is greater than or equal to the second remainder. In cases in which the first remainder is less than the second, add 11 to the first digit of the first remainder-product. The method is illustrated:

Student's part
Chosen number: 41
10-remainder: 1
11-remainder: 8

Teacher's part: by shortcut method:

$$\begin{array}{r} 11 \\ +\ 11 \\ \hline 121 \\ -\ 8 \\ \hline 41 \end{array}$$

6.03 Signs of Operation Puzzles

Fill in the proper operation sign:

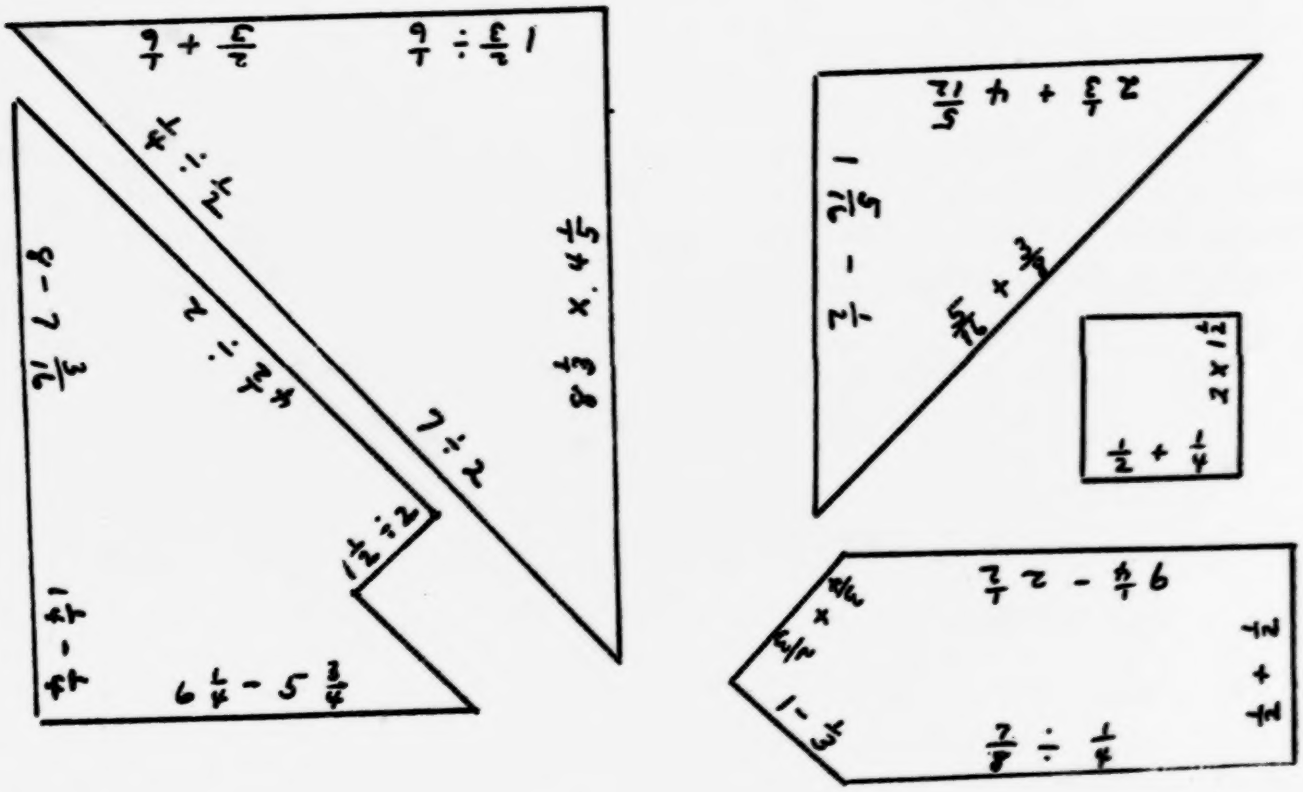
4		6		2	=	7
	▨		▨		▨	
3		3		6	=	3
	▨		▨		▨	
6		6		8	=	20
=	▨	=	▨	=	▨	=
2		3		4	=	1

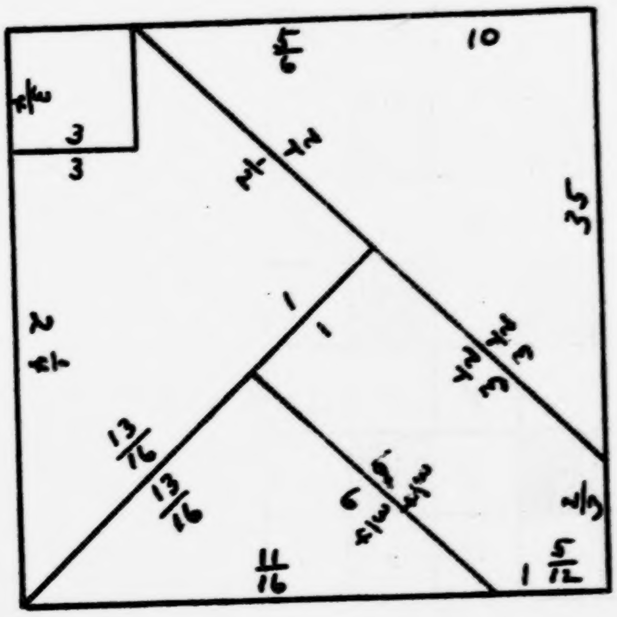
Solution:

4	+	6	÷	2	=	7
x	▨	+	▨	x	▨	x
3	x	3	-	6	=	3
÷	▨	-	▨	-	▨	-
6	+	6	+	8	=	20
=	▨	=	▨	=	▨	=
2	+	3	-	4	=	1

6.04 Fraction Puzzles

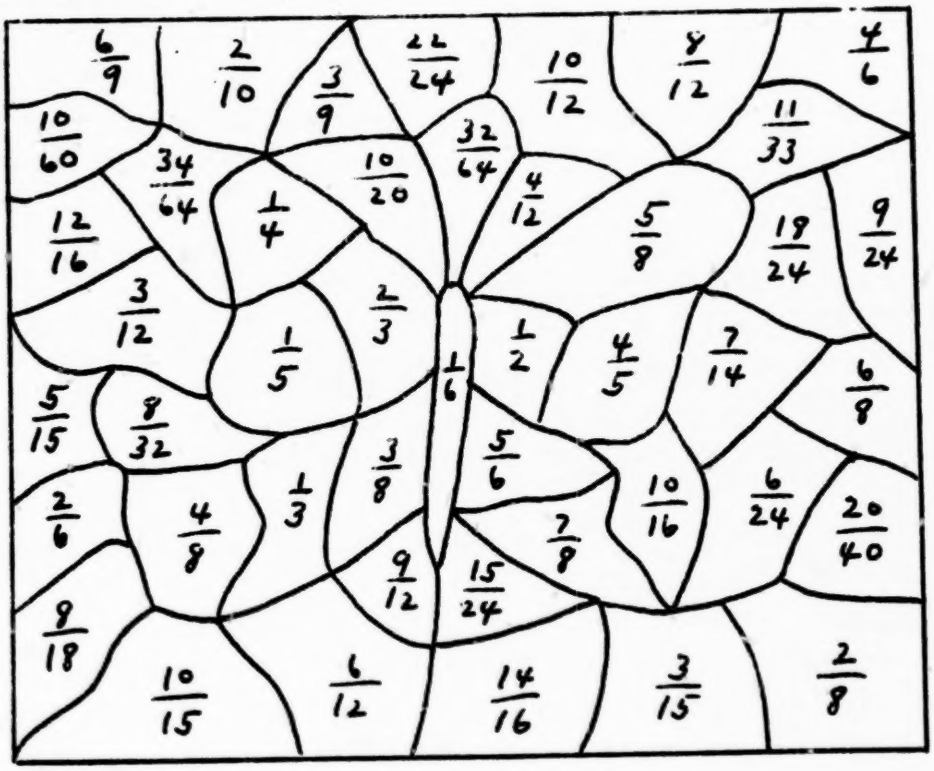
The five pieces in this puzzle will form a perfect square if they are cut out and put together correctly. The puzzle can be solved quickly by finding the answers to the number questions along the edges. When all the answers are found, match equal answers on different pieces of the square, and the puzzle will be solved.


6 1/4 - 5 3/4
1 1/2 ÷ 2
7 ÷ 2
1/2 + 1/4
7/8 ÷ 1/4


5/6
10
3/3
3
1
1
13/16
13/16
11/16
6
1 5/12

6.05 Fraction Fun

Shade all the parts having fractions in lowest terms:


6/9
2/10
3/9
22/24
10/12
8/12
4/6
10/60
34/64
10/20
32/64
11/33
12/16
1/4
4/12
5/8
19/24
9/24
3/12
2/3
1/5
1/6
1/2
4/5
7/14
5/15
8/32
6/8
3/8
5/6
2/6
4/8
1/3
10/16
6/24
20/40
7/8
9/18
9/12
15/24
10/15
6/12
14/16
3/15
2/8

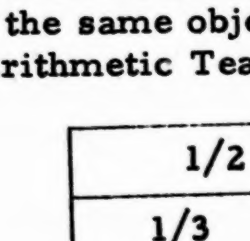
6.06 Fraction Board

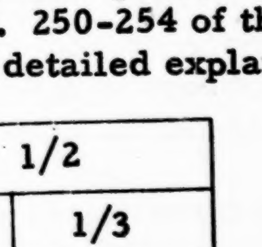
Directions for making a fraction board are given on p. 143 of the C-2 Bulletin: This board can be used to show equivalent fractions, simple addition and subtraction of fractional parts, and relative sizes of fractions of the same object. See pp. 250-254 of the December, 1957 issue of The Arithmetic Teacher for a detailed explanation of its use.

<table>
<tr><td colspan="30">1/2</td><td colspan="30">1/2</td></tr>
<tr><td colspan="20">1/3</td><td colspan="20">1/3</td><td colspan="20">1/3</td></tr>
<tr><td colspan="15">1/4</td><td colspan="15">1/4</td><td colspan="15">1/4</td><td colspan="15">1/4</td></tr>
<tr><td colspan="12">1/5</td><td colspan="12">1/5</td><td colspan="12">1/5</td><td colspan="12">1/5</td><td colspan="12">1/5</td></tr>
<tr><td colspan="10">1/6</td><td colspan="10">1/6</td><td colspan="10">1/6</td><td colspan="10">1/6</td><td colspan="10">1/6</td><td colspan="10">1/6</td></tr>
</table>

6.07 Fractional Parts of a Square

Use graph paper or colored paper to illustrate fractional parts of a square by shading or by using colors. Ask students to write the fraction numeral representing the shaded or colored portion.





6.11 Babylonian and Egyptian Fractions

Students may enjoy working with fractions in the manner of the ancient Egyptians and Babylonians.

The ancient Egyptians restricted their numerators to 1 except for the fraction 2/3. Notice how the following fractions were represented:

3/4 would be written as 1/2 + 1/4

5/8 would be written as 1/2 + 1/8

2/5 would be written as 1/4 + 1/10 + 1/20

Tables were used to give the value in unit fractions for denominators larger than 4.

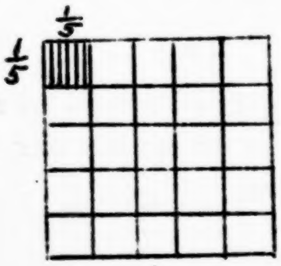
The Babylonians used sexagesimal fractions since 60 was their base. Computation with sexagesimal fractions resembles work with decimals but is more complicated because division by 60 is more difficult than division by 10.

6.12 History of Fractions

An interesting project for students would be to write a history of fractions. The Greeks considered fractions as the ratio of two numbers while the Egyptians considered a fraction as simply a part of a number. The Egyptian concept is the one that became popular in elementary work, but today modern mathematics is bringing the Greek viewpoint into favor.

It is also interesting to notice that the treatment of fractions in modern mathematics is similar to the sixteenth century treatment of fractions.

Kobel in 1514 introduced fractions by cutting an apple in parts. Tonstall in 1522 illustrated finding the product of 1/5 by 1/5 by dividing a square into 5 vertical strips and 5 horizontal strips.



Rudolff in 1530 added 2/3 and 3/4 as follows:

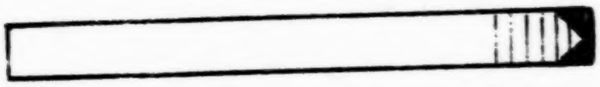
$$\begin{array}{cc} 8 & 9 \\ \hline 2/3 & 3/4 \\ \hline \multicolumn{2}{c}{12} \end{array} \quad \text{makes } 17/12$$

Chuquet in 1484 and Trenchant in 1566 explained the division of fractions by finding the lowest common denominator of both dividend and divisor and then by dividing the numerators.

6.08 Musical Straws

Soda straws may be used not only to show a relationship of mathematics to music but also to afford entertaining drill in fractions. Each student should be given 4 straws and the following directions:

Cut the first straw so that it will be 8" long. Flatten 1/2 inch of one end of the straw and cut off 1/4 inch from each corner. See the diagram below.

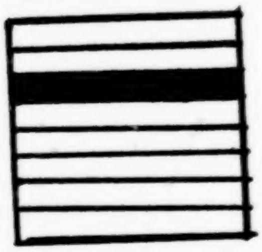


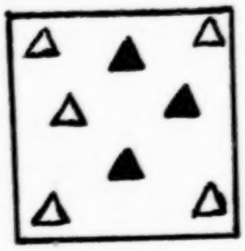
Cut the second straw so that it will be 3/4 as long, and flatten and cut as with the first straw. Cut the third straw 5/8 as long as the first and the fourth 1/2 as long as the first. Flatten and cut as before.

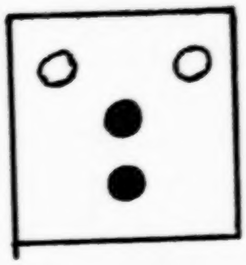
They will produce a major chord when sounded together. To play a straw, put the cut end, which is like the reed in a clarinet, in your mouth and blow. The class can be divided into four groups, with each group of students playing straws of one length. When the four groups play together, they sound a major chord. Two groups playing only the longest and shortest straws sound an octave.

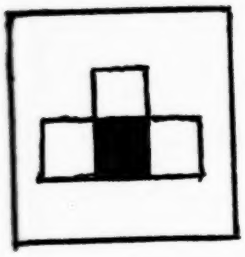
6.09 Find the Smallest Fraction

Place cards like the ones shown below on a chalk tray or on some convenient stand. Ask students to write a fraction which represents the ratio of the number of the shaded regions to the total number of regions. They may then arrange the cards to represent increasing fractional numbers. Students may be divided into teams which take turns sending a pupil to the board to select the card representing the smallest fraction until all the cards have been chosen.









6.10 Magic Squares

Magic squares may be used to give practice in addition of fractions.

$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$1\frac{1}{3}$
$\frac{1}{3}$	$1\frac{1}{4}$	$\frac{5}{6}$	$\frac{5}{12}$
$1\frac{1}{6}$	$\frac{1}{12}$	$\frac{2}{3}$	$\frac{11}{12}$
$\frac{7}{12}$	1	$1\frac{1}{12}$	$\frac{1}{6}$

$1\frac{3}{8}$	$\frac{1}{2}$	$1\frac{1}{8}$
$\frac{3}{4}$	1	$1\frac{1}{4}$
$\frac{7}{8}$	$1\frac{1}{2}$	$\frac{5}{8}$

ACTIVITIES

13 Famous Mathematicians

1768 - 1830	Fourier, a French mathematician, was the author of a book on the mathematical theory of heat conduction. This is a source book on modern methods in mathematical physics.
1777 - 1855	Gauss, the son of a day laborer in Germany, is considered by many to be one of the three greatest mathematicians of all time. He was perhaps the first to anticipate a non-Euclidean geometry, and although he did not publish his findings, he did approve the work of Bolyai and he gave the revolutionary new geometry the name of non-Euclidean. He could do complicated mental arithmetic by the time he was 3 years of age.
1789 - 1857	Cauchy, a French mathematician, contributed to the modern foundations of the calculus.
1790 - 1868	Moebius was the first to introduce homogeneous coordinates which became the accepted tool for the algebraic treatment of projective geometry. He is considered one of the founders of the modern science of topology. The Moebius strip, a surface with only one side, is easy to make.
1793 - 1856	Lobachevsky, a Russian, was perhaps the first to publish a really systematic treatment of non-Euclidean geometry.
1796 - 1893	Jacob Steiner, who had a quite ordinary career until he was thirty-eight years old, became one of the leading geometers in Germany. It is interesting to note that he did not learn to read and write until he was fourteen and that as a boy he was considered dull.
1802 - 1860	Bolyai, a dashing young army officer and son of a mathematician, also constructed a non-Euclidean geometry which was similar to Lobachevsky's. In 1832 it was published as an appendix to a mathematical book by his father.
1805 - 1865	Hamilton, an Irish mathematician, discovered quaternions and devoted most of his later life to their study. In a flash of insight he perceived the notion that an algebra where multiplication was non-commutative was possible.
1811-- 1832	Galois, the son of a small town mayor near Paris, was killed in a duel at the age of 21. The sixty pages he feverishly wrote on the evening before he was killed contained his ideas on the theory of groups and furnished the key to modern algebra.

UNIT VI

EXERCISES

1. Represent the following:

   a. the set of integers
   b. the set of all fraction numerals which represent the number 3/5

2. Solve the following:

   a. 4 + 4 ÷ 4
   b. 3 x 4 + 2 x 8
   c. 2 x (6 - 4)
   d. 8 - 3 x 2
   e. Would you rewrite any to avoid errors in solving?

3. Solve this problem by three methods:

   3/7 ÷ 1/4

4. Perform the operations if the following ordered pairs represent rational numbers:

   a. (2, 3) + (5, 6)
   b. (3, 4) - (2, 3)
   c. (3, 4) ÷ (6, 5)
   d. (3, 4) x (2, 3)
   e. (6, 1) ÷ (2, 3)

EXERCISES (answers) **UNIT VI**

1. a. $\{\ldots -3, -2, -1, 0, +1, +2, +3, \ldots\}$
   b. $\{3/5,\ 6/10,\ 9/15, \ldots\}$

2. a. 5
   b. 28
   c. 4
   d. 2
   e. Perhaps a, b, and d.

3. $$3/7 \div 1/4 = \frac{3/7}{1/4} = \frac{3/7 \times 1}{1/4 \times 1} = \frac{3/7 \times 4/4}{1/4 \times 7/7} = \frac{12/28}{7/28} \times 1 = \frac{12/28}{7/28} \times 28/28 = 12/7 = 1\ 5/7$$

$$3/7 \div 1/4 = \frac{3/7}{1/4} \times 1 = \frac{3/7}{1/4} \times 28/28 = 12/7 = 1\ 5/7$$

$$3/7 \div 1/4 = \frac{3/7}{1/4} \times 1 = \frac{3/7}{1/4} \times \frac{4/1}{4/1} = 12/7 = 1\ 5/7$$

$$3/7 \div 1/4 = 3/7 \times \frac{1}{1/4} = 3/7 \times 4/1 = 12/7 = 1\ 5/7$$

$$3/7 \div 1/4 = 3/7 \times 4/1 = 12/7 = 1\ 5/7$$

4. a. 9/6 = 1 1/2
   b. 1/12
   c. 5/8
   d. 1/2
   e. 9

ASSIGNMENT

1. □ ÷ △ = 48. What would be the answer if

   a. □ is tripled?
   b. △ is doubled?
   c. △ is halved?
   d. □ is halved?
   e. If △ is a natural number other than 0 or 1, which is larger □ or 48?
   f. If △ is a proper fraction which is larger □ or 48?

2. Bill's grandfather if 6 1/2 times as old as his grandson, who is 7 1/3 times as old as his little sister who is 1 1/2 yr. old. How old is Bill's grandfather?

3. Change:

   a. .046 unit to tenths; hundredths.
   b. 3.82 tenths to units; hundredths.
   c. 46 units to tens; hundreds.

4. First solve the following problems and then rewrite them so they will be more easily understood

   a. 5 + 5 ÷ 5
   b. 6 - 4 x 3 + 2
   c. 24 ÷ (6+6)
   d. 18 ÷ 3 - 24 ÷ 12
   e. 24 ÷ 8 ÷ 2
   f. 4 + 4 x (4 - 4)

5. What happens to the value of a fraction if

   a. the numerator is doubled?
   b. the denominator is halved?
   c. the numerator is halved and the denominator is doubled?
   d. the numerator and denominator are both doubled?
   e. the numerator is increased by twice its value?
   f. the numerator and denominator are both increased by 2? (examine carefully)

6. A man rides a bicycle 3/4 of a mile in 3 minutes. Find his speed in miles per hour.

7. If apples are selling at 3 for ten cents, find

   a. the cost of 3 dozen apples.
   b. how many can be bought for 40 cents.

8. Show that multiplication of rational numbers is commutative. Use the fractions a/b and c/d.

9. a. Change the following fractions to a common denominator and arrange from smallest to largest.

   3/15, 2/9, 1/4, 2/12

9. b. Change the above fractions to a common numerator and arrange from smallest to largest.

10. What is the multiplicative inverse of each of the following?

a. 4	d. -7
b. 2/3	e. 0
c. π	f. 1

11. What is the additive inverse of each of the following?

a. 4	d. -7
b. 2/3	e. 0
c. π	f. 1

12. Complete the following table:

	No. pair	Operation	Result	Explanation
a.	(2, 3)	+	5	2 + 3
b.	(3, 4)	−	____	____
c.	(4, □)	____	2	____
d.	(7, 3)	÷	____	____
e.	(3, 8)	____	24	____
f.	(□ , 10)	____	1/5	____

13. Perform the operations if the following ordered pairs represent rational numbers:

a. (1, 2) + (4, 5)	e. (3, 4) + (4, 3)
b. (2, 3) x (7, 4)	f. (2, 3) − (1, 5)
c. (4, 5) ÷ (2, 3)	g. (4, 1) ÷ (1, 4)
d. (4, 1) x (1, 4)	h. (4, 1) − (1, 4)

JUST FOR FUN

14. Use four 8's and represent the numbers from 1 - 10.

15. In activity 6.08 a science correlated activity may be devised. The ratio of vibrations of the diatonic scale are given in any science book as 1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, and 2. (a) Using any given length as 4 in. compute the lengths to represent each note. (b) The ratios 1, 5/4, 3/2, and 2 represent the major chord. What are these lengths if 4" is used? (c) Try making a musical straw. If you have difficulty making music, ask any child how it is done.

ASSIGNMENT (answers)

1. a. 144
   b. 24
   c. 96
   d. 24
   e. □
   f. 48

2. 71 1/2 years

3. a. .46; 4.6
   b. .382; 38.2
   c. 4.6; .46

4. a. 6; 5 + (5 ÷ 5)
   b. -4; 6-(4 x 3)+2
   c. 2; 24/(6 + 6) or leave as it is
   d. 4; 18/3 - 24/12
   e. 1 1/2; (24 ÷ 8)/2
   f. 4; 4 + 4·(4-4)

5. a. doubled
   b. doubled
   c. equal to 1/4 original value
   d. remains the same
   e. tripled
   f. increased if proper fraction and decreased if improper fraction

6. 15 mi/hr.

7. a. $1.20
   b. 12

8.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

$$= \frac{ca}{db}$$

$$= \frac{c}{d} \times \frac{a}{b}$$

9. a. 3/15 = 36/180 2/12, 3/15, 2/9, 1/4
   2/9 = 40/180
   1/4 = 45/180 (If changed to a common denominator, the greater
   2/12 = 30/180 the numerator the greater the fraction)

   b. 3/15 = 6/30 2/12, 3/15, 2/9, 1/4
   2/9 = 6/27
   1/4 = 6/24 (If changed to a common numerator, the smaller
   2/12 = 6/36 the denominator, the greater the fraction)

10. a. 1/4
    b. 3/2
    c. $1/\pi$
    d. -1/7
    e. 0 has no multiplicative inverse
    f. 1

11. a. -4 d. 7
    b. -2/3 e. 0
    c. $-\pi$ f. -1

12.

a.	(2, 3)	+	5	2 + 3
b.	(3, 4)	−	-1	3 - 4
c.	(4, 2)	÷	2	4 ÷ 2
d.	(7, 3)	÷	2 1/3	7 ÷ 3
e.	(3, 8)	x	24	3 x 8
f.	(2, 10)	÷	1/5	2 ÷ 10

13. a. (13, 10)
    b. (14, 12) or (7, 6)
    c. (12, 10) or (6, 5)
    d. (4, 4) or (1, 1)
    e. (25, 12)
    f. (7, 15)
    g. (16, 1)
    h. (15, 4)

14. 1 = 88/88 or 8/8 + 8 - 8
    2 = 8/8 + 8/8
    3 = (8 + 8 + 8) /8 or 88/8 - 8
    4 = (8 x 8) / (8 + 8)
    5 = $\sqrt{8 + 8}$ + 8/8 or 8- $\sqrt{8 + 8/8}$
    6 = 8 − (8 + 8) ÷ 8
    7 = (8 x 8 - 8) ÷ 8
    8 = 8 + 8·(8 - 8)
    9 = (8 + 8 x 8)/8 or 8/.8 - 8/8
    10 = (88 - 8) ÷ 8 or 88/8.8
    11 = 8/.8 + 8/8
    12 = (8 + 8)/(.8 - 8)

15. a. 4" 4 1/2", 5", 5 1/3", 6", 6 2/3", 7 1/2", 8"
    b. 4", 5", 6", 8"
    c. Construction

UNIT VII

OUR NUMBER SYSTEM: AN OVERVIEW

UNIT VII. OUR NUMBER SYSTEM: AN OVERVIEW

> *"I should like to have it explained," said the Mock Turtle. "Why", said the Dodo, "the best way to explain it is to do it."*
>
> *-----Lewis Carroll*

INTRODUCTION

In this unit operations performed on numbers represented by fraction numerals and by decimal numerals will be compared. It will be shown that an infinite non-repeating decimal represents a new number--the irrational, and that the union of the set of rational numbers and the set of irrational numbers is the set of real numbers. The complex numbers will be introduced as an unordered set of numbers.

CHANGING FROM FRACTION NUMERALS TO DECIMAL NUMERALS

In most computational work, decimal notation offers an advantage over fraction notation. Decimal notation is merely an extension of our decimal system of numeration and not a new system.

Consider the following examples:

.4 = 4/10
.32 = 32/100 which equals 3/10 + 2/100
.006 = 6/1000 which equals 0/10 + 0/100 + 6/1000

In each of the examples the denominator is a power of ten. This is the decimal notation and the dot is called a decimal point only in base ten. In base two .1 written in fraction notation as 1/10 has the value of 1/2 in the decimal system. Similarly, $.11_{two}$ would be written in fraction notation in base two as 1/10 + 1/100 but would equal 1/2 + 1/4 or 3/4 in the decimal system. What would $.101_{two}$ equal in the decimal system?

To change 2/5 to a decimal, solve the proportion 2/5 = n/10; n = 4. Then write 2/5 = 4/10 or .4. In a similar manner 2/5 can be changed to the decimal .40 by solving 2/5 = n/100. Thus .4 and .40 may be considered as names for the same number unless they represent measurements. Except in measurements, annexing zeros to the right of the last decimal digit does not change the number even though it changes the numeral.

The results in the example above may also be obtained by dividing 2 by 5.

ADDITION AND SUBTRACTION OF DECIMALS

Numbers expressed in decimal notation are commonly designated as "decimals" and are more conveniently added or subtracted if the numerals are written in vertical columns with the decimal points in a line so that the ones, the tenths, the hundredths, etc., are in columns.

The relationship between addition and subtraction of numbers expressed first as fraction numerals and then as decimal numerals is shown below:

Add:

$$\begin{array}{r} 4\frac{26}{100} \\ 5\frac{68}{100} \\ \underline{8\frac{43}{100}} \\ 17\frac{137}{100} = 18\frac{37}{100} \end{array} \qquad \begin{array}{r} 4.26 \\ 5.68 \\ \underline{8.43} \\ 18.37 \end{array}$$

Subtract:

$$\begin{array}{r} 8\frac{50}{100} \\ \underline{4\frac{47}{100}} \\ 4\frac{3}{100} \end{array} \qquad \begin{array}{r} 8.50 \\ \underline{4.47} \\ 4.03 \end{array}$$

MULTIPLICATION OF DECIMALS

Exercises such as 1/10 x 2/10 = 2/100 and .1 x .2 = .02 may be used to help students understand that products represented by decimal numerals are equivalent to the same products represented by fraction numerals. Such exercises may also help students discover that when numbers represented by decimal numerals are multiplied, the number of decimal places in the product equals the sum of the number of decimal places in the multiplier and the multiplicand. The parallel between multiplication of numbers expressed by fraction numerals and by decimal numerals is further illustrated:

$$4\frac{2}{10} \times 3\frac{4}{10} = \frac{42}{10} \times \frac{34}{10} = \frac{1428}{100} = 14\frac{28}{100}$$

$$\begin{array}{r} 4.2 \\ \times\, 3.4 \\ \hline 16\ 8 \\ 126\ 0 \\ \hline 14.28 \end{array}$$

DIVISION OF DECIMALS

Exercises such as 1/100 ÷ 1/10 = 1/10 and .01 ÷ .1 = .1 may be used to help students understand that quotients represented by decimal numerals are equivalent to the same quotients represented by fraction numerals. Such exercises may also help students discover that when numbers represented by decimal numerals are divided, the number of decimal places in the quotient equals the difference of the number of decimal places in the dividend and the divisor. The parallel between division of numbers expressed by fractions and by decimal numerals is further illustrated:

$$24\frac{624}{1000} \div 4\frac{32}{100} = \frac{24624}{1000} \times \frac{100}{432} = \frac{57}{10} = 5\frac{7}{10}$$

$$\begin{array}{r|l} & \overline{\quad 5.7} \\ 4.32 & 24.624 \\ & \underline{21\ 60} \\ & 3\ 024 \\ & \underline{3\ 024} \end{array}$$

The arrows are used to indicate that

$$\frac{24.624}{4.32} \times \frac{100}{100} = \frac{2462.4}{432}$$

PERCENTS AS FRACTION AND DECIMAL NUMERALS

A topic closely related to decimal fractions is percentage. "Percent" means "by the hundred" and the symbol "%" merely replaces "1/100". Many authorities believe that the traditional practice of "moving the decimal point" is confusing to students. Since % means hundredths, the "percent numeral" may be more easily understood if it is changed first to a fraction numeral and then to a decimal numeral as illustrated:

$$25\% = \frac{25}{100} = .25$$

$$110\% = \frac{110}{100} = 1.10$$

$$\frac{1}{2}\% = \frac{\frac{1}{2}}{100} = \frac{\frac{1}{2}}{100} \times \frac{10}{10} = \frac{5}{1000} = .005$$

$$\text{Similarly } .25 = \frac{25}{100} = 25\%$$

$$1.10 = \frac{110}{100} = 110\%$$

$$.005 = \frac{5}{1000} = \frac{.5}{100} = \frac{\frac{1}{2}}{100} = \frac{1}{2}\%$$

USING RATIOS TO SOLVE PERCENTAGE PROBLEMS

Most percentage problems can be placed in one of "three cases" which typify practically all such problems. These are:

Case 1: Finding a percent of a number: 25% of 60
Case 2: Finding what percent one number is of another:
6 is ______% of 8
Case 3: Finding a number when a percent of it is known:
4 is 2% of ______

It is possible to solve such problems by using proportions (equal ratios) which make it unnecessary to distinguish among cases. All three cases will be similar in structure if the idea of ratio (comparison of two numbers by division) is used. The solution depends upon the definition of equality of two ratios:

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

For example if 3/4 = 6/8, then 3 x 8 = 4 x 6. If 2n/5 = 12/15, then $2n \cdot 15 = 5 \cdot 12$ or 30n = 60 and n=2. This is verified by showing that 4/5 = 12/15.

This concept may be used to solve the three examples:

	Proportion:	Solution:
Case 1:	$\frac{25}{100} = \frac{n}{60}$	$100n = 1500$ $n = 15$
Case 2:	$\frac{6}{8} = \frac{n}{100}$	$8n = 600$ $n = 75$
Case 3:	$\frac{4}{n} = \frac{2}{100}$	$2n = 400$ $n = 200$

RATIONAL NUMBERS EXPRESSED BY REPEATING DECIMALS

In the previous chapter a rational number was defined as a number expressed by a/b or (a, b) with a and b always integers and $b \neq 0$. Any rational number may also be expressed in decimal notation.

Examples: 1/2 = .50000···   1/3 = .33333···

2/5 = .40000···   13/99 = .131313···

The examples in both columns are infinite repeating decimals. Those in the first column when written as .5 and .4 are sometimes called terminating decimals. In the decimal expansion of every rational number one digit or one group of digits repeats. For .5 and .25 the repeating digit is 0 and for .1313··· the repeating group of digits is 13.

Every rational number may be represented as an infinite repeating decimal. Conversely, every infinite repeating decimal may be represented as a rational number.

Show that .131313··· is a rational number.

Let n = .131313···
100n = 13.131313···
99n = 13 (100 n - n = 99n)
n = 13/99

Usually 1/2 is considered equivalent to the decimal .5; however, all so called terminating decimals and all integers have an alternate form:

3.0000··· = 2.9999···
.5000··· = .4999···

Show that .4999··· is a rational number.

Let n = .4999···
100n = 49.999···
10n = 4.999···

90n = 45 (100 n - 10n = 90n)
n = 1/2

That 1.0000··· = .9999··· may seem reasonable from the following examples:

$1/3 = .3333\cdots$ $2/3 = .6666\cdots$ $3/3 = .9999\cdots$

$1/12 = .0833\ldots$ $6/12 = .4999\ldots$ $12/12 = .9999\cdots$

ANOTHER TYPE OF NUMBER--THE IRRATIONAL

Not every infinite decimal is a repeating decimal. For example, a non-repeating decimal can be represented:.10100100010000···

Numbers such as $5\sqrt{2}$, $\sqrt[4]{3}$, and π cannot be expressed in fraction form as the ratio of two integers a/b, $b \neq 0$. However, they may be expressed as non-repeating infinite decimals. There is no pattern for finding the successive digits. Numbers expressed as infinite non-repeating decimals belong to the set of irrational numbers.

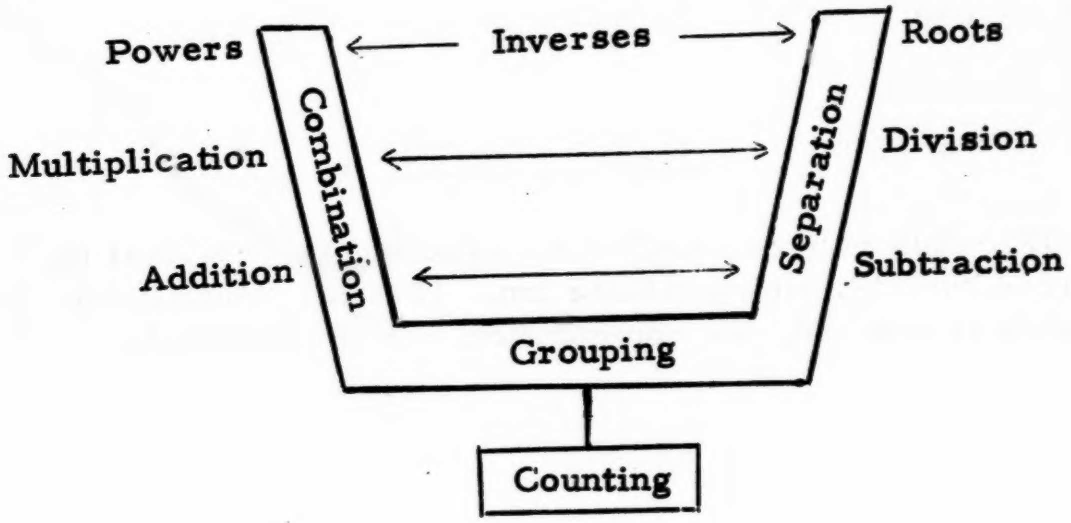
POWERS AND ROOTS

You will recall from Unit II that $10^3 = 10 \cdot 10 \cdot 10$. The numeral 3 indicates the number of times 10 is to be used as a factor. The 3 is called the exponent, 10 is called the base, and 10^3 or 1000 is called the power.

The inverse operation of finding the power is finding the root. To find the square root of a number n is to find a number k such that k x k = n. If n is the square of a rational number, then k will be rational; for example, $\sqrt{25} = 5$. If n is not the square of a rational number, then k will be irrational and can only be approximated; for example, $\sqrt{2} \approx 1.414$. The symbol $\approx$ means "approximately equals".

To find the cube root of a number n is to find a number k such that k x k x k = n. If n is the cube of a rational number, then k will be rational; for example $\sqrt[3]{64} = 4$. If n is not the cube of a rational number, then k will be irrational; for example, $\sqrt[3]{2} \approx 1.259$.

The following diagram summarizes the operations and their inverses that have been discussed in these units.



APPROXIMATING AN IRRATIONAL NUMBER EXPRESSED AS A SQUARE ROOT

Irrational numbers may be approximated by rational numbers. To approximate an irrational number expressed as a square root, a method adapted to use on a desk calculator may be used instead of the usual algorithm. This method is illustrated by two examples.

Example: Find $\sqrt{2}$

Divide 2 by some positive number less than 2 and find the average of the quotient and divisor. Then divide 2 by this average and again find the average of the resulting quotient and divisor. Continue until the desired approximation is found.

	Average of quotient and divisor:
$2 \div 1 = 2$	$\frac{2 + 1}{2} = 1.5$
$2 \div 1.5 \approx 1.333$	$\frac{1.500 + 1.333}{2} \approx 1.416$
$2 \div 1.416 \approx 1.412$	$\frac{1.416 + 1.412}{2} = 1.414$

$\sqrt{2} \approx 1.414$. The symbol $\approx$ is read "is approximately equal to".

That $\sqrt{2} \approx 1.414$ may be verified by squaring 1.414; that is, $1.414 \times 1.414 = 1.999396$, which differs from 2 by .000604.

Example: Find $\sqrt{1275}$

The square root of a larger number should first be estimated to save work in computation. Since $30^2 = 900$ and $40^2 = 1600$, then $30 < \sqrt{1275} < 40$. In this case a good estimate of the root is 35.

	Average of quotient and divisor:
$1275 \div 35 \approx 36.428$	$\frac{35.000 + 36.428}{2} = 35.714$
$1275 \div 35.714 \approx 35.700$	$\frac{35.714 + 35.700}{2} = 35.707$

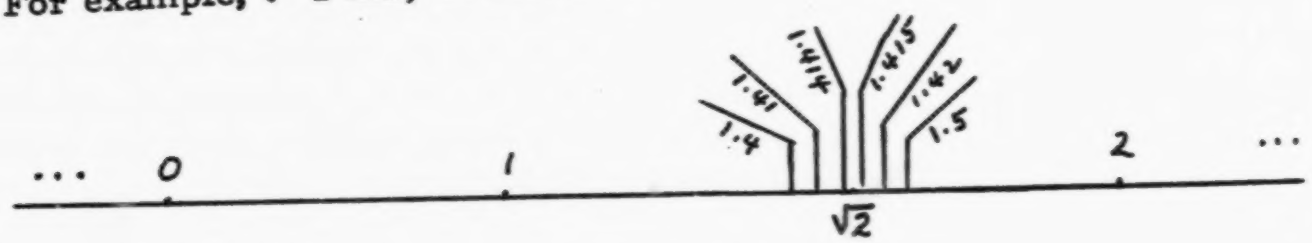
$\sqrt{1275} \approx 35.707$. This may be verified by squaring 35.707; that is, $35.707 \times 35.707 = 1274.989849$, which differs from 1275 by .010151. If a closer approximation is desired, the computation may be extended.

LOCATING IRRATIONAL NUMBERS ON THE NUMBER LINE

Examine the real number line to see how the irrational numbers may be represented on it. Consider $\sqrt{2}$ which approximates 1.4142. It is between the rational numbers 1 and 2, and its limit can be approximated by a method such as the following:

$$1.4 < \sqrt{2} < 1.5$$
$$1.41 < \sqrt{2} < 1.42$$
$$1.414 < \sqrt{2} < 1.415$$

Every irrational number can be described as the intersection of a set of nested intervals whose length approaches zero as a limit. For example, $\sqrt{2}$ may be located:



While $\sqrt{2}$ and other irrational numbers cannot be expressed as the ratio of two integers, lengths representing their values can be constructed. The $\sqrt{2}$ can be constructed with straight edge and compass because it is equal in length to the diagonal of a one-inch square.

REAL NUMBERS

The set of rational numbers and the set of irrational numbers may be considered as forming a new set of numbers called the set of real numbers.

$$\{\text{Rational numbers}\} \cup \{\text{Irrational numbers}\} = \{\text{Real numbers}\}$$

The set of real numbers like the set of integers, the set of rational numbers, and the set of irrational numbers is an ordered set. If a and b are real numbers, either $a = b$ or $a < b$ or $b < a$.

It can be shown that a real number can be assigned to every point on a number line and, conversely, that every point on the line represents a real number. This geometric representation of the real numbers is the real number line.

The rational numbers are said to be dense. Between every two rational numbers there is another rational number. To show this, select two rational numbers and find their average. For example, find a rational number between .0010 and .0010. If $1/2\,(.0100 + .0010) = .0055$, $.0010 < .0055 < .0100$.

The real numbers, like the rational numbers, are said to be dense since between every two real numbers there is another real number.

It can be shown in the theory of sets that most points on the real number line represent irrational numbers.

Between any two rational numbers there is an irrational number. This can be illustrated by selecting two repeating decimals and showing a non-repeating decimal between them.

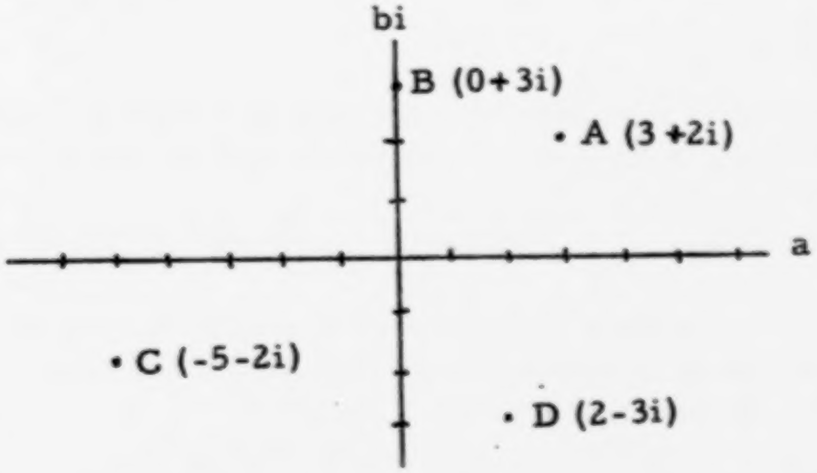
Both .1111111··· and 222222··· are rational. The irrational number .212112111··· is only one of many numbers between the two given rational numbers.

$$.111111\cdots < .2121121112\cdots < .222222\cdots$$

It can be shown that the real numbers have closure under the operations of addition, subtraction, multiplication, and division if division by 0 is excluded. The operations of addition and multiplication are commutative and associative, and multiplication is distributive over addition and subtraction. Zero is the identity element for addition and one is the identity element for multiplication. For each real number there is an additive inverse, and for each real number except 0 there is a multiplicative inverse.

COMPLEX NUMBERS (Optional)

The real number line can be considered complete with the inclusion of the irrationals. It seems reasonable to assume, however, that other numbers not represented on the real number line may be represented as at A, B, C, and D in the diagram.



There exists no real number x such that $x^2 = {}^{-}1$. However, in both theory and in practice as in applied electronics numbers such as $\sqrt{{}^{-}1}$ and $\sqrt{{}^{-}25}$ are needed. Thus a new type of number is defined:

$$i = \sqrt{{}^{-}1}$$

These numbers, which cannot be represented on the real number line, are called pure imaginary numbers and may be represented on a vertical line called the imaginary number line or imaginary axis. The horizontal line as before represents the real number line or real axis.

Numbers not on either axis may be represented by the general form

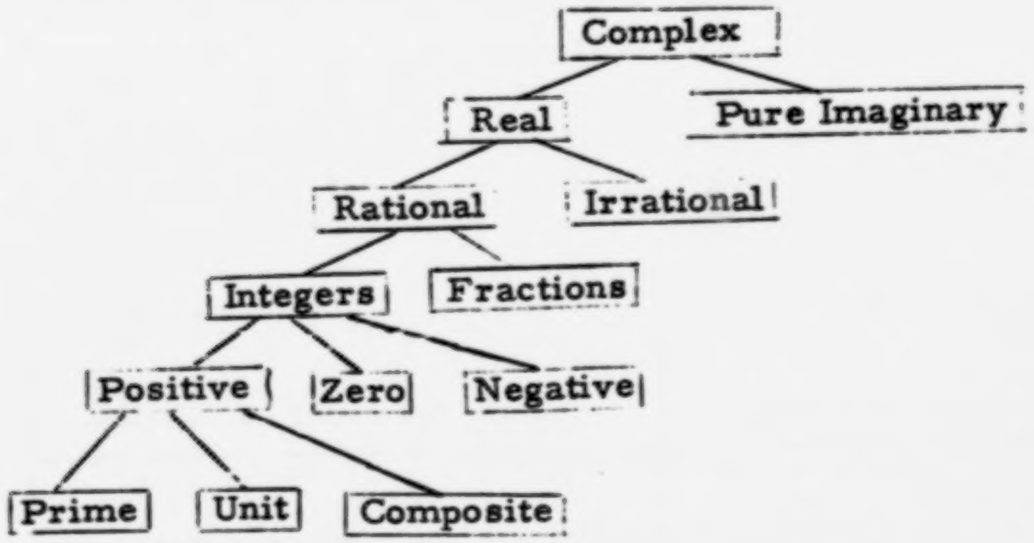
$a + bi$ and are called complex numbers; for example, on the diagram A is the point that represents the complex number $3 + 2i$. If $b = 0$, then $a + bi$ represents a real number; if $a = 0$ and $b \neq 0$, then $a + bi$ represents a pure imaginary number. Thus the real numbers and the pure imaginary numbers may be considered as subsets of the set of complex numbers.

It can be shown that the complex numbers have closure under the operations of addition, subtraction, multiplication, and division if division by 0 is excluded. The operations of addition and multiplication are commutative and associative, and multiplication is distributive over addition and subtraction. Zero is the identity element for addition and one is the identity element for multiplication. For each complex number there is an additive inverse, and for each complex number except 0 there is a multiplicative inverse. Note that these same principles are assumed for the real numbers.

Although the set of real numbers is an ordered set, the set of complex numbers is not an ordered set. "Greater than" and "less than" are not defined for complex numbers. Recall that for any real numbers a and b either $a = b$, $a > b$, or $a < b$.

The number system can be extended further. However, these extensions are in advanced work. For the purposes of elementary mathematics, the number system extended to the complex numbers may be considered complete.

DIAGRAM OF THE COMPLEX NUMBER SYSTEM



OPERATIONS WITH COMPLEX NUMBERS FORMALLY DEFINED (Optional)

The complex numbers $a + bi$ and $c + di$ may be represented as ordered pairs (a, b) and (c, d) of real numbers and the following definitions given for the operations of addition, subtraction, multiplication, and division: Note that $i \cdot i = i^2 = {}^-1$.

Operation	Definition	Illustration	Interpretation
Addition	$(a, b) + (c, d) =$ $(a + c,\ b + d)$	$(2, 3) + (4, 5) =$ $(6, 8)$	$(2 + 3i) + (4 + 5i) =$ $6 + 8i$
Subtraction	$(a, b) - (c, d) =$ $(a - c,\ b - d)$	$(2, 3) - (4, 5) =$ $({}^{-}2,\ {}^{-}2)$	$(2 + 3i) - (4 + 5i) =$ ${}^{-}2 + {}^{-}2i$
Multiplication	$(a, b) \times (c, d) =$ $(ac - bd,\ ad + bc)$	$(2, 3) \times (4, 5) =$ $({}^{-}7, 22)$	$(2 + 3i)\ (4 + 5i)$ $= (2+3i)4+(2+3i)5i$ $= 8+12i + 10i - 15$ $= (8-15) + (12i+10i)$ $= -\ 7 + 22i$
Division	$(a, b) \div (c, d) =$ $\left(\frac{ac + bd}{c^2 - d^2}, \frac{bc - ad}{c^2 - d^2} \right)$	$(2, 3) \div (4, 5)$ $= \left(\frac{8+15}{16-25}, \frac{12-10}{16-25} \right)$ $= \left(\frac{23}{-9}, \frac{2}{-9} \right)$ $= \left(\frac{{}^{-}23}{9}, \frac{{}^{-}2}{9} \right)$	$(2 + 3i) \div (4 + 5i)$ $= \frac{(2+3i)}{(4+5i)} \cdot \frac{(4-5i)}{(4-5i)}$ $= \frac{8-10i+12i+15}{16 - 25}$ $= \frac{23 + 2i}{{}^{-}9}$ $= \frac{{}^{-}23}{9} + \frac{{}^{-}2}{9}\ i$

SUMMARIZATION OF PRINCIPLES FOR THE FOUR OPERATIONS

The principles of closure, commutativity, and associativity for the four operations may be summarized:

CLOSURE

Sets	+	−	×	÷
Natural Numbers	Yes	No	Yes	No.
Integers	Yes	Yes	Yes	No
Rationals	Yes	Yes	Yes	Yes*
Real Numbers	Yes	Yes	Yes	Yes*
Complex Numbers	Yes	Yes	Yes	Yes*

*if division by zero is excluded

COMMUTATIVITY

Sets	+	−	X	÷
Natural Numbers	Yes	No	Yes	No
Integers	Yes	No	Yes	No
Rational Numbers	Yes	No	Yes	No
Real Numbers	Yes	No	Yes	No
Complex Numbers	Yes	No	Yes	No

ASSOCIATIVITY

Sets	+	−	X	÷
Natural Numbers	Yes	No	Yes	No
Integers	Yes	No	Yes	No
Rational Numbers	Yes	No	Yes	No
Real Numbers	Yes	No	Yes	No
Complex Numbers	Yes	No	Yes	No

REFERENCES FOR TEACHERS

UNIT VII

Adler--Mathematics, Exploring the World of Number & Space: p. 26, 28-29
33-35, 39-45

Banks--Elements of Mathematics: pp. 205-217, 222-232

Danzig--Number, The Language of Science: pp. 82-91, 101-121, 261

Johnson & Glenn--The Pythagorean Theorem: pp. 2-44
The Number Patterns: pp. 2-40

Larson--Oddities of Numbers, Grade 7

Merrill--Mathematical Excursions: pp. 44-61

NCTM, 24th Yearbook--Growth of Mathematical Ideas, K-12: pp. 39-42, 52-61

NCTM, 25th Yearbook--Instruction in Arithmetic: pp. 249-269

Osborn--Extending Mathematics Understanding: pp. 87-96

Swain--Understanding Arithmetic: pp. 155-191

ACTIVITIES

7.01 Fradey Percey

This game , played with cards, provides drill in fractions, decimals, and percentage. The deck of cards usually has 54 cards although the number may vary according to the amount of drill and review needed. Some cards are illustrated below:

1/2	50%	.50
1/4	25%	.25
7/8	87½%	.875

Any number from 2 to 6 may play.

The cards are shuffled and dealt one at a time. The object of the game is to get matching triples--fractions, decimals, and percent . When a player has three matching cards, he lays them down face upward on the table. If a player makes a mistake, the first child spotting the mistake collects all three cards laid down.

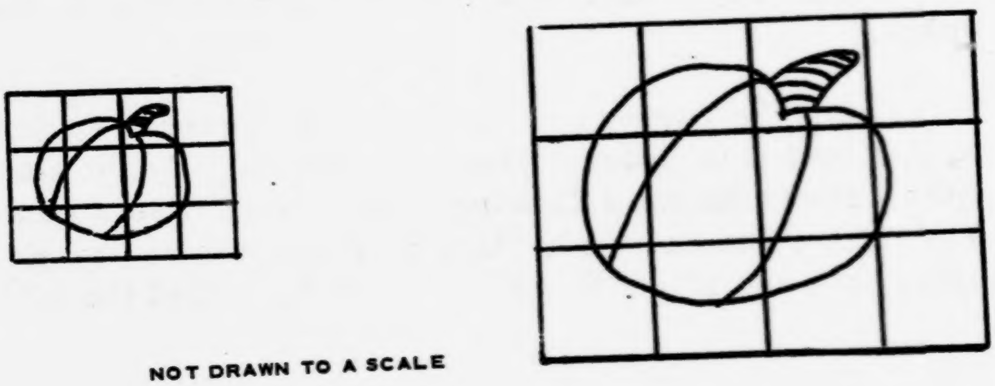
The person to the left of the dealer begins the play. He draws one card from the person to his right, and then lays down any matching triples. If he cannot put any down, the person to his left draws a card from him, etc.

The game ends when one player is out of cards. Each set of matching triples counts 10 points and each card left in a player's hand at the end counts 2 points against him. The one with the highest score wins.

7.02 Copying by Squares

This exercise should lend understanding to work with ratios. Several days before the lesson ask each child to bring a simple cartoon or comic strip to school.

Distribute 9 x 12 graph paper to the students. Have the students rule the small comic or cartoon picture into 1/4 inch squares.



NOT DRAWN TO A SCALE

ACTIVITIES

Divide the graph paper into one-inch squares. Then copy the picture by squares on the large sheet of paper. In this illustration the ratio is 1:4.

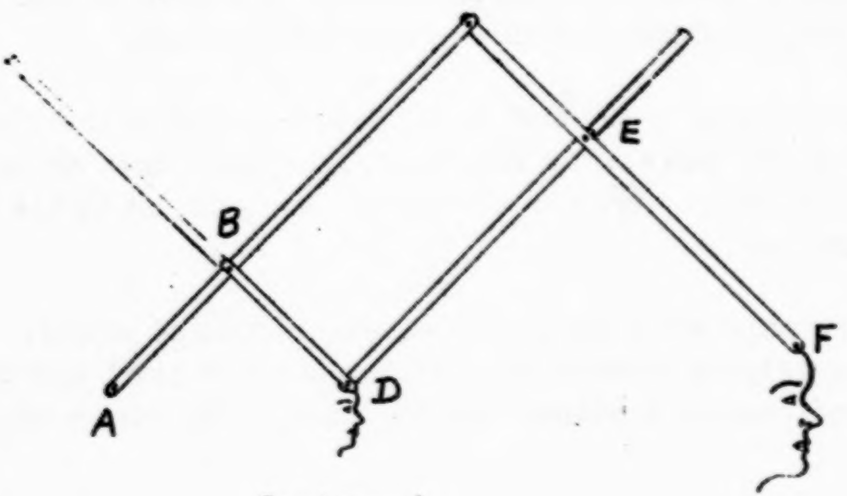
The student may measure the length of his paper and compare it with the picture and use perhaps a 1:5 or a 1:3 or 1:2 or even a 1:1 ratio.

The child will be surprised at his own art work.

7.03 Pantograph

Materials needed to construct a pantograph will include 1/2" - 1" strips of plastic, metal or wood, and paper fasteners or flat rivets.

Construct the pantograph as illustrated below making AC and CF = 9", BD = 3", DE = 6". Locate B and E by letting AB and CE equal 3".



Pantograph

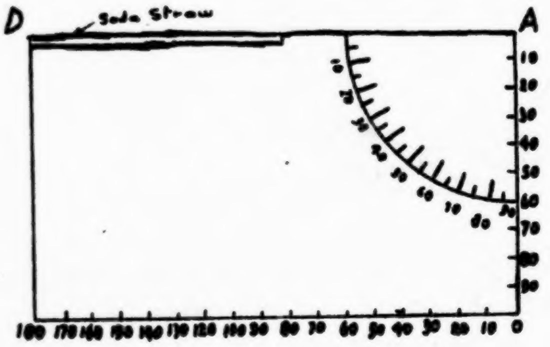
Observe the ratio of AC:AB and of CF:CE. To construct a pantograph to make a figure five times as tall, change the ratio of AC:AB and CF:CE from 3:1 to 5:1.

7.04 Hypsometer-Clinometer

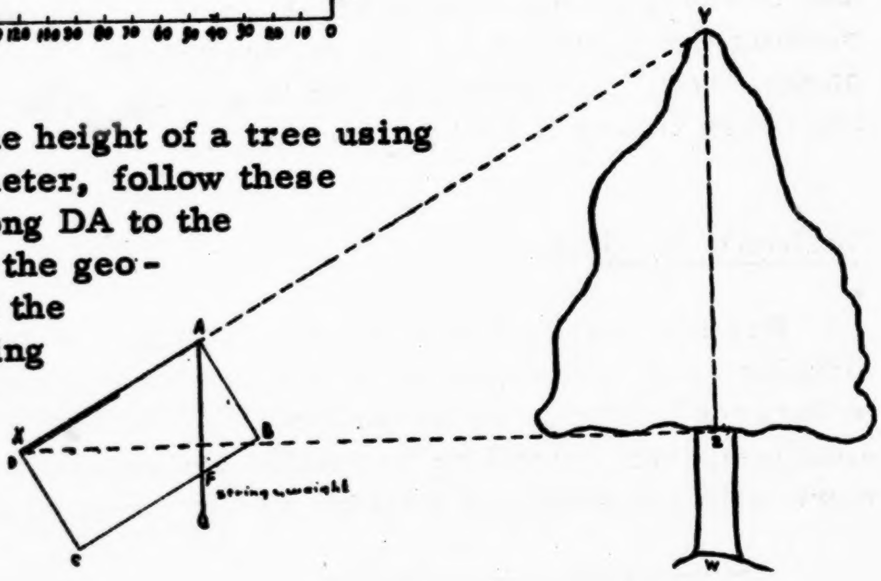
See p. 149 of C-2 Bulletin for construction or follow the directions below.

Label the corners of a sheet of 10 squares to the inch-graph paper A, B, X and D in order. Mark 10, 20, 30, •••100 along the sides AB and BC as shown on the following page. With A as a center, draw an arc of 90° with a radius of 5". Use a protractor and mark off the divisions along this arc 5°, 10°, 15°, ••• 85°. Label the 10°, 20°, 30°, ••• 80°

divisions. Fasten the paper on the cardboard either using glue or masking tape. Using a needle insert a 12" string at A and fasten on the back side with masking tape or Scotch tape. On the front side, fasten a fishline sinker or some weight on the end of the string. Tape a soda straw along DA. This becomes the sighting device.


D
Soda Straw
A
10
20
30
40
50
60
70
80
90
10 20 30 40 50 60 70 80 90
180 170 160 150 140 130 120 110 100 90 80 70 60 50 40 30 20 10 0

To figure out the height of a tree using a hypsometer-clinometer, follow these directions. Sight along DA to the top of the tree. Use the geometric principle that the ratios of corresponding sides of similar triangles are in proportion.


Y
A
X
D
B
Z
F
string & weight
C
W

The angle of elevation is YXZ. Triangle ABF if similar to triangle YXZ. If AB = 100 and BF = 40 on the chart, and if the distance XZ is measured in feet by a tape, then ZY may be computed by using the proportion: AB: BF = XZ: ZY.

Add WZ to ZY to find the total height of the tree.

7.05 Percentage Lotto

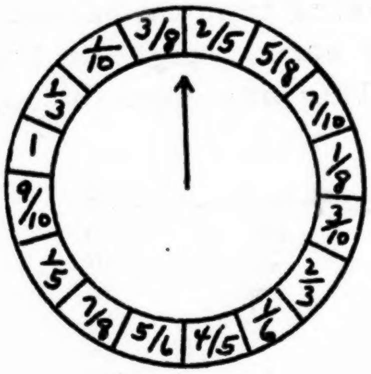
Cards can be constructed as the two below showing different percents.

$83\frac{1}{3}$%	70%	75%
10%	$12\frac{1}{2}$%	100%
$62\frac{1}{2}$%	$87\frac{1}{2}$%	20%

25%	$33\frac{1}{3}$%	$83\frac{1}{3}$%
30%	20%	$66\frac{2}{3}$%
$87\frac{1}{2}$%	40%	$16\frac{2}{3}$%

ACTIVITIES

A spinner like the one below is used.



One child may act as leader and as he spins, he records the fraction and corresponding % on a small card. As he calls the fraction, those playing the game cover the corresponding % with buttons, corn, pieces of paper, etc. The first student to get his card covered wins. A check by the class to see if he has covered the card correctly affords a good review.

7.06 Decimals by Size

Prepare cards for students like the one below. Each child draws a similar card and represents the numbers in order of size from smallest to largest in each row by writing "1" for the smallest, "2" for the next smallest, etc., until he has filled the duplicate card. He may check his work with the answers written on the reverse side.

0.5%	.001	0.04	0.08
.82	36%	10%	.2
101%	1	1.2	99%
150%	201%	1.6	2

Front

4	3	1	2
2	1	3	4
1	4	2	3
3	2	4	1

Reverse side

7.07 Simon Stevin and Decimals

Read about Simon Stevin (1548-1620), the first writer to give a systematic treatment of decimal fractions. He was a native of Bruges and later lived in the Netherlands. He called a whole number a unit and gave it the symbol ⓪ ; the tenth part of an integer he called a prime. and gave it the symbol ① ; and the tenth part of a unit prime he called a second and gave it the symbol ② . His first book on decimals called

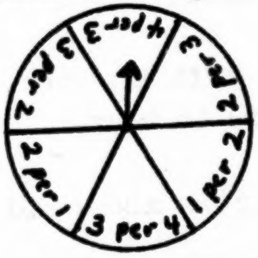
ACTIVITIES

"La Thiende" was published in Leyden, Holland, in 1585. One of the problems in multiplication looks like this:

```
        ⓪①②
        3257
        8946
      ------
       19542
      13028
     29313
    26056
    --------
    29137122
      ⓪①②③④
```

7.08 Spin the Ratio

Construct a spinner like the one shown below:



Either cards for each child may be made with the fractions given below on each square, or the teacher may write a group on the board.

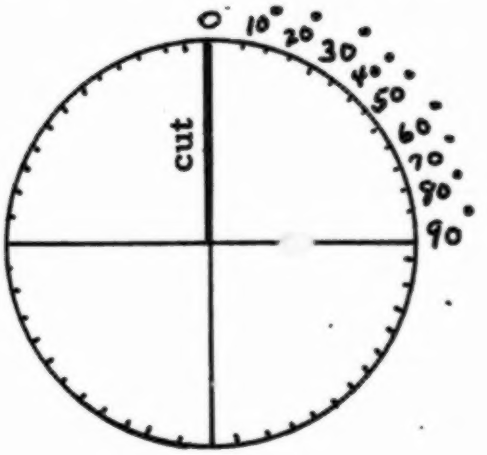
$\frac{32}{24}$	$\frac{6}{12}$	$\frac{18}{12}$
$\frac{12}{8}$	$\frac{9}{12}$	$\frac{18}{9}$
$\frac{21}{28}$	$\frac{24}{36}$	$\frac{100}{75}$

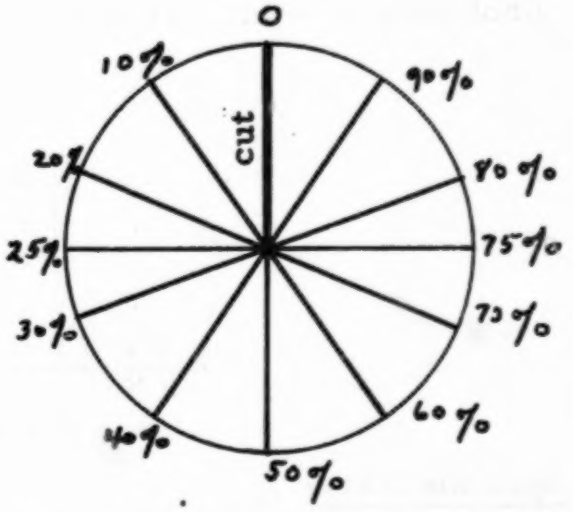
If the cards are used and a game played, the first student covering his card with corn, buttons, etc., is the winner.

If the teacher writes a group on the board, teams may be chosen; when a ratio is determined by the spinner, one member of Team A goes to the board and draws a line through every fraction that it may represent. Two points are scored for each correct attempt and 1 point deducted for each mistake made. The team with the highest score becomes the winner.

7.09 Circle Chart for Percents

Cut two circles from sheets of construction paper of different colors. Use a protractor to divide each circle into 10 sections (or 20 sectors, etc.) Write the number of degrees in each sector of one circle and write the percents in each sector on the other.



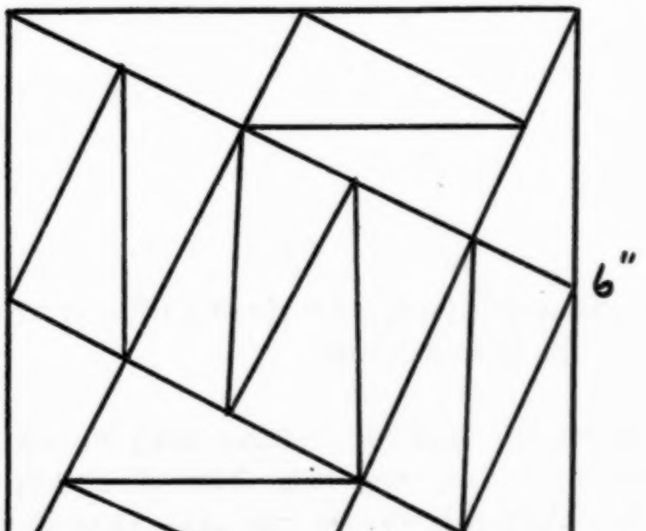


Cut along one radius of each circle and insert one circle in the other so that their centers and circumferences coincide.

Fasten both circles on a piece of cardboard with a paper fastener and adjust them so that the percentage circle is free to turn.

To use the circles, rotate the percentage circle and read the number of equivalent degrees on the other circle.

7.10 Percentage Designs



Connect the midpoints of each side of a square to the opposite corner to form 2 intersecting pairs of parallel lines which will then enclose a small inner square.

Bisect these segments and connect lines as shown in the figure at the left.

Three to six colors may be used in coloring the figure, and then the pupils may try to discover what percentage of the large square is represented by each color.

A student may cut out the small triangles to make various designs and find what percentage of the large square is being used for the design.

7.11 Pi

The irrational number π calculated to the first 50 decimal places is

3. 14159265258979323646264338327950288419716939937510582097494459230 78164 06286208998628034825342117O679. . . .

ACTIVITIES

Writing a history of π is suggested as an interesting student project. The Hebrews used 3 for the value of π . Archimedes estimated it to be between 3.140845... and 3.142857... The ancient Chinese used 3.162..., 3.1555..., and 3.1415929...

In 1607, William Jones, an English mathematician, first used the symbol π to indicate the ratio of the circumference of a circle to its diameter. However, in mathematics π has outgrown this original definition. In 1665, for example, John Wallis proved the relationship of π to a quotient of the form:

$$\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdots}{1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdots}$$

Leibniz a few years later showed the relationship of π to the infinite series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

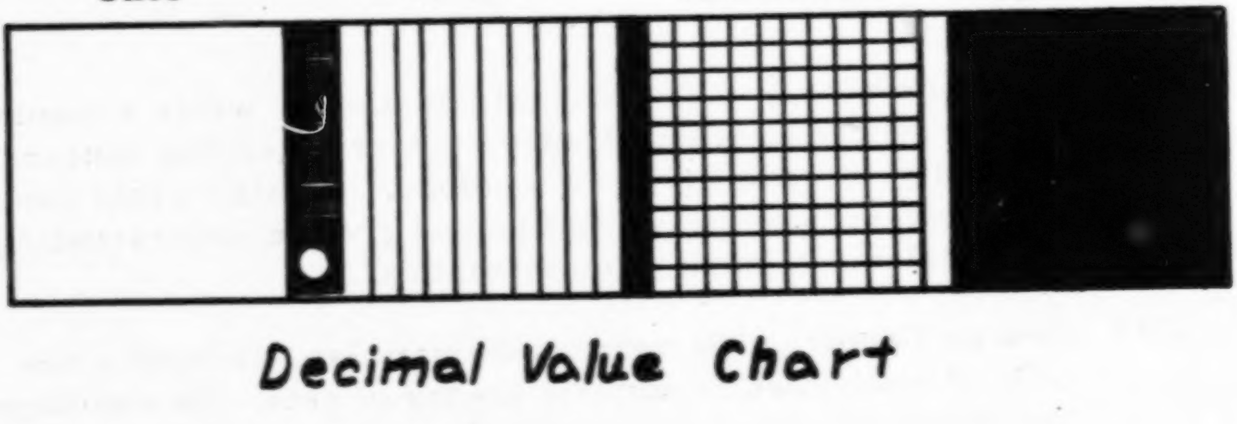
7.12 Pi and Probability

The following simple but interesting experiment can be used to approximate the value of π.

Mark off on a large sheet of paper a series of lines 2 inches apart. Break toothpicks into 1" lengths until you have 10. Next hold the sticks about a foot above the ruled paper and drop them. Count the number of sticks that either touch or cross a line. Do this 100 times. Total the number of sticks that have touched or crossed a line and divide 1000 by this total. The (total sticks tossed) divided by (total sticks crossing or touching a line) approximates π .

7.13 Decimal Value Chart

A decimal value chart may be constructed as shown in the diagram below:



Decimal Value Chart

.14 Famous Mathematicians

1814 - 1897	James Joseph Sylvester, an English mathematician, was interested in higher algebra. At one time he taught at the University of Virginia, but resigned and went back to England after an unhappy experience with one of his students. In 1877 he was called to Johns Hopkins University where, it is said, he did more than any man of his time to establish graduate work in mathematics in America.
1815 - 1897	Weierstrass, a teacher in a Prussian school, later became a professor of mathematics at the University of Berlin. He was interested in number theory and analysis. He helped establish the foundations of the calculus.
1815 - 1897	Arthur Cayley, an English mathematician and lawyer, did much of his work with Sylvester. He wrote about matrices. You may wish to find out about Cayley numbers which were named after him.
1815 - 1864	George Boole, an English mathematician, overcame early handicaps of poverty and lack of education. He showed how the laws of formal logic could themselves be made the subject of a calculus and thereby laid the foundation for the modern mathematical theory of logic. In 1854 he published the "Laws of Thought".
1826 - 1866	Bernard Riemann, the son of a German country minister, developed many non-Euclidean geometries. In the two classic ones, called single-elliptic and double-elliptic, he assumed instead of the Euclidean parallel postulate the Riemannian parallel postulate: There are no parallel lines. Albert Einstein used Riemannian geometry as the mathematical model of his theory of relativity.
1831 - 1916	Richard Dedekind, a German mathematician, constructed a rigorous theory of irrational numbers, which he defined as cuts.
1845 - 1912	Henry Poincaré, a French mathematician, wrote a number of papers that vitally influenced modern theories concerning relativity, probability, and topology. He also wrote popular and semi-popular books that helped give an understanding of the problems of modern mathematics.
1845 - 1918	George Cantor, a German mathematician, created a new field of mathematics with his theory of sets. He developed the theory of transfinite cardinal numbers based on a mathematical treatment of infinite sets.

EXERCISES

1. Arrange the following in order of magnitude from smallest to largest:

   a. .14, .146, .1392, .02154
   b. 33%, 1/3, 3/10, 33.4%
   c. 7/9, 7/8, 7/11, 7/4

2. Change each of the following repeating decimals to their fractional equivalents:

   a. .141414...
   b. .090909...
   c. .125125...

3. Which of the following expressions represent rational numbers?

   a. .1515... c. .1234135... e. 13/29 g. $\sqrt{2}$

   b. $\sqrt{36}$ d. π f. 1.414 h. .888...

4. Locate points representing the complex numbers; then write as ordered pairs.

   a. $3 + 4i$ c. 3 e. $-i$

   b. $-2-5i$ d. $4-i$

5. Given the ordered pair (2, 3)

   a. What integer may they represent?
   b. What rational number may they represent?
   c. What complex number may they represent?
   d. Locate a point on a graph if this ordered pair represents the coordinates of a point (x, y).
   e. Would the ordered pair (3, 2) represent the same integer? the same rational number? the same complex number?

6. Find an approximation for each of the following:

   a. $\sqrt{3}$

   b. $\sqrt{200}$

EXERCISES (answers)

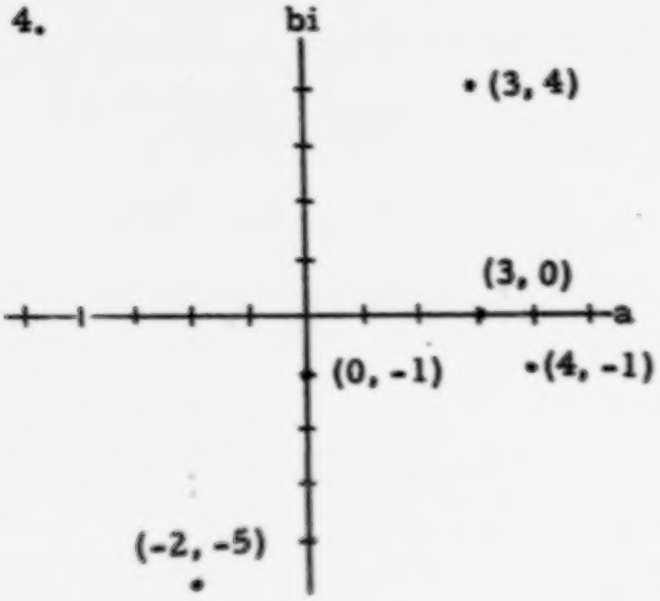
UNIT VII

1. a. .02154, .1392, .14, .146
   b. 3/10, 33%, 1/3, 33.4%
   c. 7/11, 7/9, 7/8, 7/4

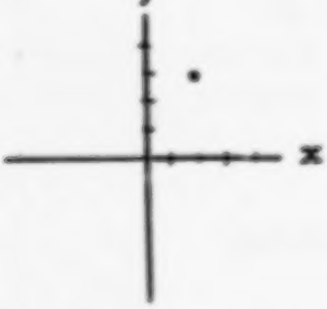
2. a. 14/99
   b. 1/11
   c. 125/999

3. a, b, e, f, h

4.



5. a. $^{-}1$
   b. 2/3
   c. 2 + 3i
   d.



   e. No; $^{+}1$. No; 3/2. No; 3 + 2i.

6a. $1 < \sqrt{3} < 2$

$3 \div 1 = 3$

$\frac{3+1}{2} = 2$

$3 \div 2 = 1.5$

$\frac{2.00 + 1.50}{2} = 1.75$

$3.000 \div 1.750 \approx 1.714$

$\frac{1.750 + 1.714}{2} = 1.732$

$\sqrt{3} \approx 1.732$

6b. $14 < \sqrt{200} < 15$

$200 \div 14 \approx 14.28$

$\frac{14.00 + 14.28}{2} = 14.14$

$200 \div 14.14 \approx 14.144$

$\frac{14.140 + 14.144}{2} = 14.142$

$\sqrt{200} \approx 14.142$

ASSIGNMENT

UNIT VII

1. Arrange the following in order of magnitude from smallest to largest:

   a. .34, .342, .338, .3146
   b. 11/8, 11/4, 11/9, 11/16
   c. 38%, 3/8, .374, 37.2%

2. Change each of the following repeating decimals to their fraction equivalents:

   a. .030303...
   b. .121212...
   c. .214214214...
   d. .015015015...

3. Which of the following common fractions will have 0 as the repeating digit?

   a. 8/25
   b. 1/7
   c. 7/20
   d. 7/15
   e. 4/13
   f. 9/27

4. Which of the following expressions represent rational numbers?

   a. π
   b. .222...
   c. 3/4
   d. 3.14
   e. 15/51
   f. $\sqrt[3]{8}$
   g. 50
   h. .2134176...

5. Find the value of each of the following if expressed in base ten notation.

   a. $.101_2$

   b. $.32_4$

   c. $.24_5$

6. Consider the number 2.

   a. Is it a natural number?
   b. Is it a prime number?
   c. Is it an integer?
   d. Is it a rational number?
   e. Is it a real number?
   f. Is it a complex number?

7. Consider the number 3/7.

   a. Is it a natural number?
   b. Is it an integer?
   c. Can it be expressed as a repeating decimal?
   d. Is it a rational number?
   e. Is it a real number?
   f. Is it a complex number?

ASSIGNMENT

8. Given the ordered pair of integers ($^{+}2$, $^{-}3$).

   a. What is the result of the binary operation of addition on them? of subtraction? of multiplication? of division?
   b. What rational number do they represent?
   c. What complex number do they represent?
   d. Locate a point on a graph if they represent the coordinates (x, y) of a point.

9. A merchant advertised as follows: "Prices reduced 100% on all items." What would he be doing if this sign were correct?

10. Billy had 2 dimes and a nickel and found a nickel and a penny. Later he lost a nickel.

    a. What per cent gain did his findings represent?
    b. What per cent decrease did his loss represent?

11. Solve the following using the ratio method explained in this unit.

    a. The average rent in a certain town 10 years ago was $50. Since then rents have increased 120%. What is the average rent today?

    b. Bill Smith earned $100 a week. He was given a 10% increase in salary. One year later business was poor and his salary was cut 10%. What was his weekly salary then?

    c. George Johnson also earned $100 a week. His salary was cut 10%. At the end of the next year it was increased 10%. What was his weekly salary then?

12. Draw a real and an imaginary axis and locate the following points; then represent them as ordered pairs:

a. $3 - i$	c. $-3i$	e. $-3-i$
b. $-1+i$	d. $4 + 2i$	f. 2

13. If you can construct $\sqrt{2}$ by measuring the length of the diagonal of a 1" square, how could you construct the $\sqrt{3}$ using a similar method?

14. Find an approximation for each of the following:

    a. $\sqrt{5}$

    b. $\sqrt{300}$

ASSIGNMENT (answers)

1. a. .3146, .338, .34, .342
   b. 11/16, 11/9, 11/8, 11/4
   c. 37.2%, .374, 3/8, 38%

2. a. 1/33
   b. 4/33
   c. 214/999
   d. 5/333

3. a, c

4. b, c, d, e, f, g

5a. $.101_2 = 1/2 + 0/2^2 + 1/2^3 = 1/2 + 1/8 = 5/8$ or $.625_{10}$

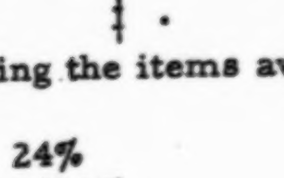
   b. $.32_4 = 3/4 + 2/4^2 = 3/4 + 2/16 = 14/16 = 7/8 = .875_{10}$

   c. $.24_5 = 2/5 + 4/5^2 = 2/5 + 4/25 = 14/25 = .56_{10}$

6. a. yes
   b. yes
   c. yes
   d. yes
   e. yes
   f. yes

7. a. no
   b. no
   c. yes
   d. yes
   e. yes
   f. yes

8. a. -1; +5; -6; -2/3
   b. -2/3
   c. 2 - 3i
   d.



9. Giving the items away

10. a. 24%
    b. 16.1%

11. a. $110
    b. $99
    c. $99

a. 220/100 = n/50

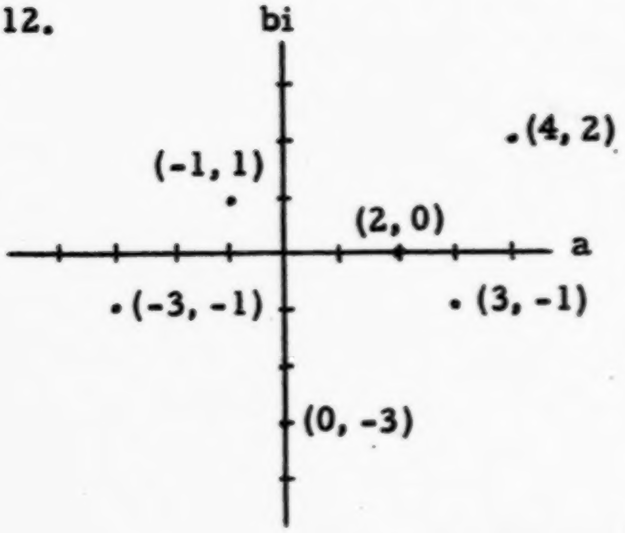
b. 100% + 10% - 11% = 99%

$$\frac{n}{100} = \frac{99}{100}$$

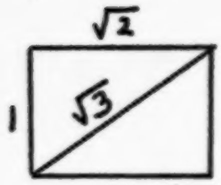
c. 100% - 10% + 9% = 99%

$$\frac{n}{100} = \frac{99}{100}$$

12.



13.



By constructing the diagonal of a rectangle whose sides are $\sqrt{2}$ and 1.

14a. $2 < \sqrt{5} < 3$

$5 \div 2 = 2.5$

$$\frac{2.00 + 2.50}{2} = 2.25$$

$5.00 \div 2.25 \approx 2.222$

$$\frac{2.250 + 2.222}{2} = 2.236$$

$\sqrt{5} \approx 2.236$

14b. $17 < \sqrt{300} < 18$

$300 \div 17 \approx 17.6470$

$$\frac{17.0000 + 17.6470}{2} = 17.3235$$

$300 \div 17.3235 \approx 17.3175$

$$\frac{17.3235 + 17.3175}{2} = 17.3205$$

$\sqrt{300} \approx 17.3205$

UNIT VIII

OTHER INTERESTING FACTS ABOUT NUMBERS

UNIT VIII. OTHER INTERESTING FACTS ABOUT NUMBERS

> *"Work consists of whatever a body is obliged to do, and play of whatever a body is not obliged to do."*
>
> -----*Mark Twain*

INTRODUCTION

In this unit tests of divisibility and some interesting topics from number theory that are included in elementary mathematics will be discussed. Such questions as these will be answered: What is the difference between the expression that a number is divisible by two and the expression that a number may be divided by two? Why can twelve blocks be arranged in several rectangular patterns while thirteen blocks cannot be so arranged? What are amicable numbers? What are perfect numbers? What is a finite number system?

DIVISIBILITY OF NATURAL NUMBERS

A natural number x is said to be divisible by a natural number y ($y \neq 0$) if $x \div y = k$, where k is a natural number and there is no remainder. Thus 8 is divisible by 2, but 9 is not divisible by 2.

Simple tests can be used to determine divisibility of some numbers. Some of these divisibility tests will be examined for numbers expressed in base ten notation.

DIVISIBILITY BY 2

A number is divisible by 2 if the number represented by the last digit of its numeral is divisible by 2. To speak of even numbers as being divisible by 2 is repetitious because even numbers are defined as those numbers divisible by 2.

To understand the test for divisibility by 2, represent a general number in base 10 by $1000a + 100b + 10c + d$. Since 10, 100 and 1000 are always divisible by 2, $1000a + 100b + 10c$ will always be divisible by 2. Therefore, only the divisibility of d need be considered.

Example: 4258 is divisible by 2 because 8 is divisible by 2.

DIVISIBILITY BY 3 OR BY 9

A number is divisible by 3 if the number represented by the sum of its digits is divisible by 3, and by 9 if the number represented by the sum of its digits is divisible by 9.

To understand this rule, represent a general number by $1000a + 100b + 10c + d$, which can be rewritten as:

$$(999 + 1)a + (99 + 1)b + (9 + 1)c + d = (999a + 99b + 9c) + (a + b + c + d)$$

Since 999, 99, and 9 are all divisible by 3 and by 9, then 999a, 99b, and 9c are divisible by both 3 and 9. Hence the number will be divisible by 3 if (a + b + c + d), representing the sum of the digits, is divisible by 3, and divisible by 9 if (a + b + c + d) is divisible by 9.

Example: 4254 is divisible by 3 because 4 + 2 + 5 + 4 = 15, which is divisible by 3.

Example: 146, 826 is divisible by 9 because 1 + 4 + 6 + 8 + 2 + 6 = 27, which is divisible by 9.

DIVISIBILITY BY 4

A number is divisible by 4 if the number represented by the last 2 digits of its numeral is divisible by 4.

Since 100 is divisible by 4, any higher multiple of 100 will be divisible by 4. For a general number represented by 1000a + 100b + 10c + d, 1000a + 100b will always be divisible by 4. Hence the number will be divisible by 4 if (10c + d) is divisible by 4.

Example: 10532 is divisible by 4 since 32 is divisible by 4.

DIVISIBILITY BY 5

A number is divisible by 5 if the last digit of its numeral is 5 or 0. The base in the decimal system of numeration is 10 and 10 and all higher powers of 10 are divisible by 5. Therefore, only a number represented by the units digit need be considered when testing for divisibility by 5.

For a general number represented by 1000a + 100b + 10c + d, 1000a + 100b + 10c is divisible by 5. If d is divisible by 5, then the number is divisible by 5. Of the 10 digits only 0 and 5 represent numbers that are divisible by 5. Hence the number is divisible by 5 only if d equals 5 or 0.

Example: 1425 is divisible by 5.

DIVISIBILITY BY 6

A number is divisible by 6 if the number represented by the last digit of its numeral is divisible by 2 and if the sum of the digits is divisible by 3. Since 6 is the product of 2 and 3, the tests for divisibility by both 2 and 3 must hold.

Example: 4296 is divisible by 6 because 6 is divisible by 2 and 4 + 2 + 9 + 6 = 21, which is divisible by 3.

DIVISIBILITY BY 7

There is no easy test for divisibility by 7 but it is interesting to examine a test that is sometimes used. To test a number for divisibility by 7, first subtract the number represented by twice the last digit of its numeral from the number represented by the remaining digits. If this difference is divisible by 7, the original number is divisible by 7. Remember that a difference of 0 is divisible by 7. If the difference is negative, the negative difference may be used or a multiple of 7 may be added to it.

If a general number is represented by $1000a + 100b + 10c + d$, and if $(100a + 10b + c) - 2d$ is divisible by 7, the original number is divisible by 7.

Observe that $(1000a + 100b + 10c + d) \div (100a + 10b + c - 2d)$ equals 10 plus a remainder of 21d.

$$\begin{array}{r} 10 \\ \underline{100a + 10b + c - 2d}\,\overline{)\,1000a + 100b + 10c + d} \\ \underline{1000a + 100b + 10c - 20d} \\ 21d \end{array}$$

If the division is correct, then by using the inverse operation of multiplication the original number equals the divisor times the quotient plus the remainder:

(1) $1000a + 100b + 10c + d = 10(100a + 10b + c - 2d) + 21d$

If $100a + 10b + c - 2d$ is divisible by 7, then it has a factor 7 and some other factor k. When 7k is substituted for $100a + 10b + c - 2d$ in (1) the expression becomes $70k + 21d$ and the distributive law can be applied to show that the expression has the factor 7. It follows that the expression equivalent to the number also has the factor 7. Hence the original number is divisible by 7 if $100a + 10b + c - 2d$ is divisible by 7.

$$\begin{aligned} 1000a + 100b + 10c + d &= 10\,(7k) + 21d \\ &= 70k + 21d = 7\,(10k + 3d) \end{aligned}$$

Example: Is 378 divisible by 7? $37 - 2(8) = 21$ and 21 is divisible by 7; therefore, 378 is divisible by 7.

DIVISIBILITY BY 8

A number is divisible by 8 if the number represented by the last three digits of its numeral is divisible by 8. The proof is similar to the one for the test of divisibility by 4 and is left for you to prove. Remember that 1000 and all multiples of 1000 are divisible by 8.

Example: 39,872 is divisible by 8 since 872 is divisible by 8.

DIVISIBILITY BY 10

A natural number is divisible by 10 if the last digit of its numeral is 0. This proof is also left for you since it is very similar to the proof for divisibility by 5.

Example: 4260 is divisible by 10 but 234 is not divisible by 10.

DIVISIBILITY BY 11

A natural number is divisible by 11 if the sum of numbers represented by the odd-numbered digits from right to left minus the sum of the numbers represented by the even numbered digits from right to left is a multiple of 11. This method is the same as combining numbers represented by the digits when taken in order from right to left with signs first plus and then minus. If the result is a multiple of 11, the number is divisible by 11. Remember that 0 is a multiple of 11.

To understand this rule, represent a general number as 1000a + 100b + 10c + d.

$$\begin{aligned}1000a + 100b + 10c + d &= (1001-1)\,a + (99+1)\,b + (11-1)c + d \\ &= 1001a - a + 99b + b + 11c - c + d \\ &= (1001a + 99b + 11c) + (-a + b - c + d)\end{aligned}$$

Observe that $(-a + b - c + d) = (d - c + b - a)$

The number represented by $(1001a + 99b + 11c)$ is divisible by 11. If $(d - c + b - a)$, which represents a number equal to the algebraic sum of the digits, designated alternately as plus and minus, is divisible by 11, the original number is divisible by 11. By agreement the units digit will be considered positive.

Example: 84,238 is divisible by 11 since 8 - 3 + 2 - 4 + 8 = 11, which is divisible by 11.

NUMBER CONGRUENCES

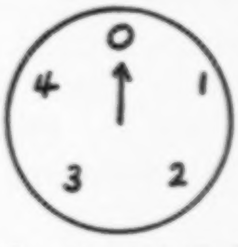
The numbers 9 and 13 have the same remainder when each is divided by 4 and are said to be congruent, modulo 4. The divisor 4 is called the modulus and "13 is congruent to 9, modulo 4" is written "$13 \equiv 9$, mod 4". This means that the difference 13 - 9 is divisible by the modulus 4.

When two numbers A and B have the same remainder when each is divided by M ($M \neq 0$), then A is said to be congruent to B, modulo M. This may be written "$A \equiv B$, mod M". This means that the difference A - B is divisible by M.

The theory of number congruence was created and developed by Gauss. It is used in number theory and finds practical elementary use in clock arithmetics, which may also be called finite arithmetics or modular arithmetics.

MODULAR OR CLOCK ARITHMETIC

How can you determine what time it will be in 15 hours if it is now 10:00 A. M.? Instead of counting 15 divisions around the clock, think "10 + 15 = 25" and "25 - 2(12) = 1". The time will be 1:00 A. M. This is an application of number congruences, modulo 12. A clock need not have 12 numerals. Consider a clock with 5 numerals like the one shown at the left. When the hand is at 2 and moves 4 spaces in a clockwise direction, it will be at 1. Then 2 plus 4 equals 1 for this clock. This may also be expressed as $2 + 4 \equiv 1$, mod 5. Likewise $4 + 3 \equiv 2$, mod 5.



In the previous unit the structure of our number system was presented. Principles of closure, commutativity, and associativity were discussed and the principle of distributivity of multiplication over addition was emphasized. The concepts of identity and inverse elements for both addition and multiplication were developed. This same structure can be demonstrated in a clock arithmetic where the modulus is a prime number. Thus clock arithmetics are often used in modern elementary courses to help students better understand the nature of a mathematical system. As an example, consider a finite arithmetic with modulus 5. First, addition and multiplication tables may be constructed from the five hour clock:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

That the principles stated for our number system also hold in a modulo 5 system may be verified completely from the tables. For instance, the 25 possible addition combinations may be listed to show closure under the operation:

0 + 0 = 0
0 + 1 = 1
0 + 2 = 2
0 + 3 = 3
0 + 4 = 4

1 + 0 = 1
1 + 1 = 2
1 + 2 = 3
1 + 3 = 4
1 + 4 = 0

2 + 0 = 2
2 + 1 = 3
2 + 2 = 4
2 + 3 = 0
2 + 4 = 1

3 + 0 = 3
3 + 1 = 4
3 + 2 = 0
3 + 3 = 1
3 + 4 = 2

4 + 0 = 4
4 + 1 = 0
4 + 2 = 1
4 + 3 = 2
4 + 4 = 3

These same examples may be used to show that addition is commutative. By listing the 25 multiplication combinations, you will observe that the system is closed under the operation of multiplication and that multiplication is commutative. Do you observe from each table that symmetry about the principal diagonal indicates commutativity?

Examples such as (2 + 4) + 3 = 2 + (4 + 3) and (2 x 4) x 3 = 2 x (4 x 3) illustrate the associative principle of addition and the associative principle of multiplication; 2 x (3 + 4) = 2 x 3 + 2 x 4 illustrates that multiplication is distributive over addition.

In this system the additive identity is zero and the multiplicative identity is one, but notice that the additive inverse of 2 is 3 and not $^{-}2$ and that the multiplicative inverse of 2 is also 3 and not 1/2. The additive inverse of 1 is 4 and the multiplicative inverse of 1 is 1. What is the additive inverse and the multiplicative inverse of 3 and of 4? It is also interesting to note that 1 - 3 = 3 and that 2 ÷ 3 = 4. This is an arithmetic with no fractional numbers and no negative numbers.

Consider now a modular arithmetic whose modulus is a composite number, e.g., 4. Will the same principles hold? Examine the addition and multiplication tables below and determine the additive and multiplicative inverses.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Again, the additive identity is 0 and the multiplicative identity is 1. Each number has an additive inverse, but observe that 2 does not have a multiplicative inverse. If the modulus is a composite number, not all the numbers will have multiplicative inverses. If the modulus is prime, a multiplicative inverse will exist for each number except 0. However, in all cases, whether the modulus is prime or composite, there will be no negative numbers and no fractional numbers in the system. Observe, too, that if the modulus is composite, the answers in multiplication are not unique since 2 x 1 = 2 and 2 x 3 = 2.

INTERESTING THEORIES CONCERNING PRIME NUMBERS

Prime numbers have interested number theorists throughout the years. Euclid proved that an infinite number of primes exist. Many questions and conjectures involving primes still remain unsolved. Some of these include the following:

1. Is every even number (other than) the sum of two primes or the sum of a prime and a unit as was conjectured by Goldbach in the 18th century?

2. Is every even number the difference of two primes or the difference of a prime number and one?

3. How can a prime number greater than a given prime number be found?

4. How can the next prime after a given prime be found?

5. How can the number of primes less than a given number be determined?

6. How can the nth prime be computed directly from a formula?

FINDING PRIME NUMBERS BY SIEVES

About 250 B.C. Eratosthenes discovered a method of locating primes, which is now known as the "Sieve of Eratosthenes" because it actually "sieves out" the primes. He merely wrote the numerals representing numbers from 1-100 and then crossed out every numeral after 2 that represented a multiple of 2, every numeral after 3 that represented a multiple of 3, every numeral after 5 that represented a multiple of 5, and every numeral after 7 that represented a multiple of 7. The remaining numerals except '1' represented prime numbers.

This method can be shortened by using the array at the left or by using the array of numerals representing odd numbers shown at the right.

Using the array representing odd numbers, begin with '3', '5' and '7' and cross out every 3rd, 5th, and 7th numeral, respectively. The remaining numerals, except '1', represent prime numbers. The even prime 2 should be added to complete the set of prime numbers between 1 and 107, inclusive.

~~1~~	2	3	~~4~~	5	~~6~~
7	~~8~~	~~9~~	~~10~~	11	~~12~~
13	~~14~~	~~15~~	~~16~~	17	~~18~~
19	~~20~~	~~21~~	~~22~~	23	~~24~~
~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~
37	~~38~~	~~39~~	~~40~~	41	~~42~~
43	~~44~~	~~45~~	~~46~~	47	~~48~~
~~49~~	~~50~~	~~51~~	~~52~~	53	~~54~~
~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~
67	~~68~~	~~69~~	~~70~~	71	~~72~~
73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~
79	~~80~~	~~81~~	~~82~~	83	~~84~~
~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~
97	~~98~~	~~99~~	~~100~~	101	~~102~~

~~1~~	3	5	7	9	11
13	~~15~~	17	19	~~21~~	23
~~25~~	~~27~~	29	31	~~33~~	~~35~~
37	~~39~~	41	43	~~45~~	47
~~49~~	~~51~~	53	~~55~~	~~57~~	59
61	~~63~~	~~65~~	67	~~69~~	71
73	~~75~~	~~77~~	79	~~81~~	83
~~85~~	~~87~~	89	~~91~~	~~93~~	~~95~~
97	~~99~~	101	103	~~105~~	107

The primes between 1 and 100 include 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

DETERMINING WHETHER A NUMBER IS PRIME OR COMPOSITE

A shorter method than the "Sieve of Eratosthenes" may be used to determine whether a number is prime or composite.

If no prime number less than or equal to the approximate square root of a given number divides the given number, then the number is prime. Generally, N is prime if no prime $\leq \sqrt{N}$ divides N.

Is 119 prime or composite? The approximate square root of 119 is 11. Since 11^2 is greater than 119 it is necessary to test 119 for divisibility by only the primes 2, 3, 5, and 7.

Tests for 2, 3, and 5 fail, but 7 divides 119. Therefore, 119 is a composite number with factors 7 and 17.

Is 229 prime or composite? The approximate square root of 229 is 15. Test 229 for divisibility by 2, 3, 5, 7, 9, 11 and 13 to determine whether it is prime or composite. Since none of these numbers divides 229, it is a prime number.

FINDING THE GREATEST COMMON DIVISOR

Even numbers are numbers that have a common factor 2. However, some numbers such as 12 and 16 may have a larger factor than 2 in common. For example, 12 equals 4 x 3, and 16 equals 4 x 4; thus 4 is a factor of both 12 and 16. Again, 50 and 20 have a common factor 10.

The largest common factor of two or more numbers is called their greatest common factor (G. C. F.) or their greatest common divisor (G. C. D.)

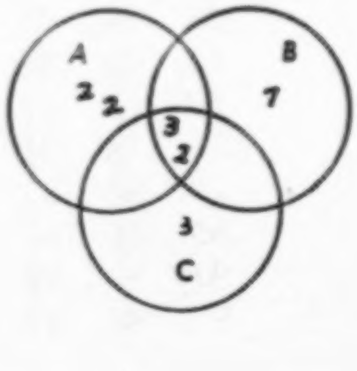
One method of finding the greatest common factor of several numbers is to decompose each number into a set of primes and find the intersection of the sets. The product of the factors in the intersection is the G. C. F. This method is illustrated in finding the greatest common factor of 24, 42, and 18.

{factors of 24} = $A=\{3, 2, 2, 2\}$

{factors of 42} = $B=\{3, 7, 2\}$

{factors of 18} = $C=\{3, 3, 2\}$

$A \cap B \cap C = \{3, 2\}$



The G. C. F. of 24, 42, and 18 is 3 x 2 or 6.

PERFECT NUMBERS

A perfect number is a number which is equal to the sum of its proper divisors. The number itself is not considered a proper divisor. The first perfect number discovered was 6. Notice that 1 + 2 + 3 = 6. The second perfect number was 28. Ancient people attached great importance to these numbers. Some attempted to explain the creation of the world in 6 days and the revolution of the moon about the earth in 28 days because 6 and 28 were perfect numbers.

By the end of the first century the perfect numbers 6, 28, 496, and 8128 had been discovered. In 1644 the perfect number 33, 550, 336 was discovered. Mersenne discovered the next 6 perfect numbers and Edward Lucas discovered the twelfth perfect number in 1876.

Five additional perfect numbers have been computed on Standards' Western Automatic Computer (SWAC) at the University of California by R. M. Robinson. More about these numbers appears in the March '53 issue of Scientific American.

AMICABLE NUMBERS

Amicable numbers were thought to symbolize friendship. These are numbers such that the proper divisors of each total the other number. For example, 220 and 284 are amicable numbers because the proper divisors of 284 (1, 2, 4, 71 and 142) total 220 and the proper divisors of 220 (1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110) total 284.

A few amicable numbers and the men who are given credit for discovering them are:

220-284	Pythagoras
17296 - 18416	Fermat
9, 363, 584 - 9, 437, 056	Descartes

Nicolo Paganini discovered 16 pairs and Euler discovered 30 pairs.

There are other interesting numbers that have attracted the attention of both mathematicians and laymen, and there will no doubt be many others. While many of these may not presently have utilitarian value, some practical application may be found later. In the meantime they, like puzzles, stimulate mathematical thought and afford entertainment.

> *At last the Dodo said, "Everybody has won, and all must have prizes."*
>
> *-----Lewis Carroll*

REFERENCES FOR TEACHERS

UNIT VIII

Adler--Exploring the World of Numbers and Space: pp. 18-20, 46-47, 85-91
The New Mathematics: pp. 90-124

Banks--Elements of Mathematics: pp. 107-115

C-2 Bulletin, Thinking in the Language of Mathematics: pp. 103-107, 117

Dantzig--Number, the Language of Science: pp. 37-57, 256-291

Hogben--The Wonderful World of Mathematics: pp. 30-39

Johnson & Glenn--Invitation to Mathematics: pp. 1-60

Larsen--Huge Numbers, Grade 7
Oddities of Numbers, Grade 7
This and That, Grade 8: pp. 12-15
Number Magic, Grade 5
Numbers Do Strange Things, Grade 5
Some Curious Numbers, Grade 6

Merrill--Mathematical Excursions: pp. 1-12, 108-113

NCTM, 24th Yearbook--Growth of Mathematical Ideas, K-12, pp. 397-402

NCTM, 25th Yearbook--Instruction in Arithmetic: pp. 121-146, 320-354

Osborn, et al, --Extending Mathematics Understanding: pp. 115-122

Smith & Ginsburg--Numbers & Numerals: pp. 39-50

Swain--Understanding Arithmetic: pp. 113-131

8.01 Number Congruences - Magic Squares - Clock Arithmetic

An interesting relationship between number congruences, magic squares, and clock arithmetic is shown below:

If the numerals for the natural numbers 1 through 9 are arrayed in three columns, the numbers represented by the numerals in each column will be congruent, modulo 3.

1	2	3
4	5	6
7	8	9

$1 \equiv 4 \equiv 7$, mod 3

$2 \equiv 5 \equiv 8$, mod 3

$3 \equiv 6 \equiv 9$, mod 3

Each element in a 3 x 3 magic square with numerals 1 - 9 is congruent to 1, 2, or 3, modulo 3, and can be replaced by '1', '2', or '3'. Since 3 is the zero element in a modulo 3 system, the '3' may be replaced by '0'.

A

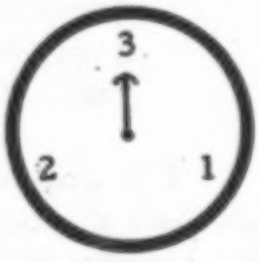
8	1	6
3	5	7
4	9	2

B

2	1	3
3	2	1
1	3	2

Each element in square A is congruent to the corresponding element in square B, mod 3. Notice that B is also a magic square.

The three hour clock shown at the right may be used to show that $1 \equiv 4 \equiv 7$, that $2 \equiv 5 \equiv 8$, and that $3 \equiv 6 \equiv 9$. Begin at 1 and count clockwise. Notice that 4 and 7 correspond to 1, that 5 and 8 correspond to 2, and that 6 and 9 correspond to 3.



When consecutive numbers are represented in arrays, those represented in each column are congruent. For example, those represented in columns on a calendar as shown below are congruent, modulo 7.

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Fig. 1

28	13	22
15	21	27
20	29	14

Fig. 2

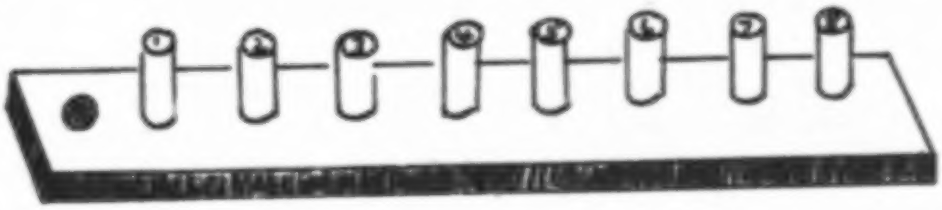
0	6	1
1	0	6
6	1	0

Fig. 3

Any square array chosen from the larger one may be rearranged to form a magic square as illustrated in Fig. 2. This magic square will in turn represent a new magic square modulo 7 as shown in Fig. 3. Remember that 6 + 1 = 0 mod 7. Larger square arrays on a calendar or on a hundred board may also be used.

8.02 Jumping Frogs

The object of this old puzzle is to move or jump the frogs in either direction so that their order is reversed from 1, 2, 3, 4, 5, 6, 7, 8 to 8, 7, 6, 5, 4, 3, 2, 1.



JUMPING FROGS

Construct the puzzle out of a convenient piece of wood about an inch wide, a half-inch thick, and 8 or 10 inches long. Cut 8 pegs of dowel stock so that the figures 1 to 8 can be written on the top as in the illustration above. Have 9 holes bored and leave the first hole empty.

Solution for 8 pegs:

jump 2, 4, 6, 8, move 7; jump 5, 3, 1; move 2;
jump 4, 6, 8, 7, move 5; jump 3, 1, 2, move 4;
jump 6, 8, 7, 5, move 3; jump 1, 2, 4, move 6;
jump 8, 7, 5, 3, move 1; jump 2, 4, 6, move 8.

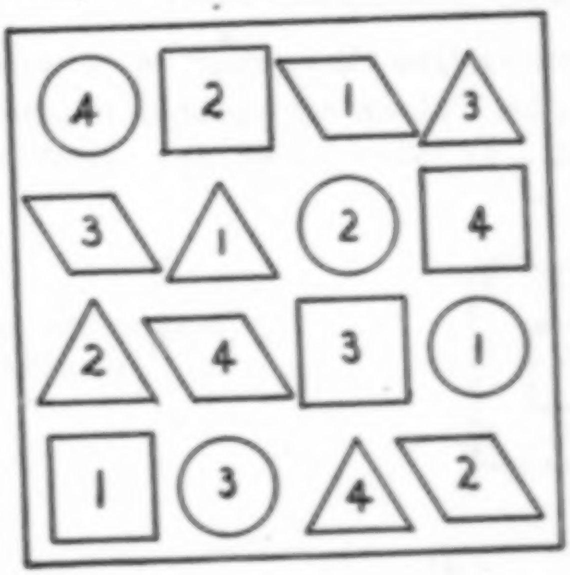
For any number, the same formula will apply--the even numbers up, and the odd numbers down, repeated half the number of times of the largest figure, plus one additional shift so that the vacant hole will be in the original position.

8.03 Clock Arithmetic and Nuclear Energy

An interesting article explaining how clock arithmetic can be used to find the percent of particles entering lead sheeting in atomic experiments is given in the December 1959 issue of "The Mathematics Teacher" pp. 604-607.

8.04 Mathematical Squares

An interesting puzzle is to arrange in a square array the numerals 1, 2, 3, 4 which are written on 4 different geometric figures or colored circles and so arranged that each numeral and each geometric figure of the same shape appear only once in each row, column, and diagonal.



Several different arrangements are possible. The 4 corners may also be required to contain different numerals and figures as shown. Colored marbles may be used instead of numerals, and 36 numerals instead of 16 may be used.

These squares have practical application in experimental agriculture where different seeds are planted in different soils. They also have practical application in laboratory tests of a similar nature in biology and chemistry.

8.05 A Reversible Magic Square

Examine this square for interesting properties. Hold it before a mirror. This is sometimes called the IXOHOXI magic square because of its reversibility.

1111	8881	8818	1188
8188	1818	1881	8111
1888	8118	8188	1811
8811	1181	1118	8888

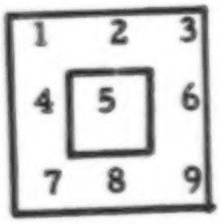
The numerals 1111, 1118, 1181, 1188, 1811, 1818, 1881, 1888, 8111, 8118, 8181, 8188, 8811, 8818, 8881, and 8888 were used to construct this magic square.

Replace the 1's with 2's and the 8's with 3's to make the entries 2222, 2223, etc. Can the resulting series be used to make a magic square?

Try other 4 digit numerals made from combinations of only two different digits.

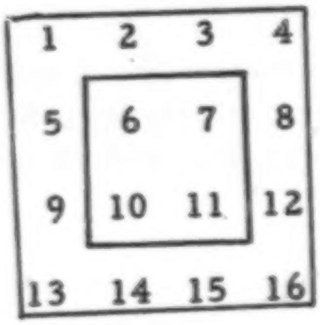
8.06 Patterns for Sums in Number Arrays

Number arrays like these may be used for some interesting discoveries in addition. Mark off squares or rectangles on a hundred board, a calendar, or make an array. Find patterns for equal sums in each one.



1	2	3
4	5	6
7	8	9

How many combinations of 10 can you find? 1 + 9 = 2 + 8, etc.

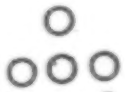


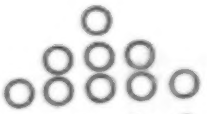
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

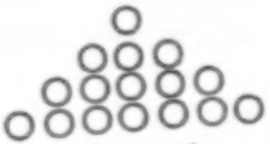
How many combinations of 17 can you find? 1 + 16 = 2 + 15, etc.

8.07 Number Patterns

In adding odd numbers a student may discover that the sum of the first two odd numbers is 4 or 2 x 2, that the sum of the first 3 odd numbers is 9 or 3 x 3, etc. This leads to the generalization $1 + 3 + 5 + \dots + 2n - 1 = n^2$.

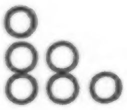


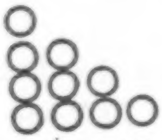




Students may wish to read about Gauss, who amazed his teachers and fellow students by his ability to solve a much more difficult progression problem at the age of 10.

The addition of consecutive numbers also follows a pattern which leads to the generalization $1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$.





8.08 Number Patterns Using Multiplication and Subtraction

By observing number patterns, students are often able to make discoveries. These patterns are formed by multiplication and subtraction:

1 x 2 = 2
2 x 3 = 6
3 x 4 = 12
4 x 5 = 20
5 x 6 = 30

30 20 12 6 2
10 8 6 4
2 2 2

0 x 4 = 0
1 x 8 = 8
2 x 12 = 24
3 x 16 = 48
4 x 20 = 80

80 48 24 8 0
32 24 16 8
8 8 8

For a start your students may like to make patterns with these:

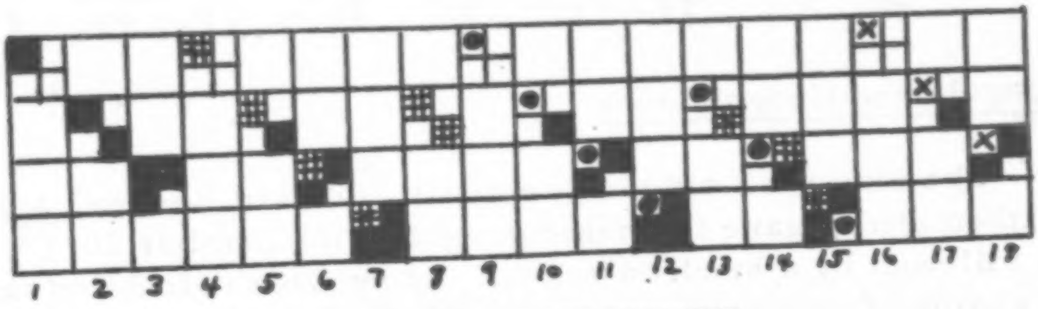
a. 0 x 3	b. 1 x 2	c. 0 x 4	d. 0 x 4
1 x 4	2 x 3	1 x 8	2 x 8
2 x 5	3 x 4	2 x 12	4 x 12
3 x 6	4 x 5	3 x 16	6 x 16
4 x 7	5 x 6	4 x 20	8 x 20

8.09 Prime Numbers

Finding out more about prime numbers may be an interesting student project. One of the largest known primes is 188, 894, 659, 785, 808, 547, 841 or $5 \cdot 2^{75} + 1$.

Euler discovered that every prime except 2 can be expressed either as 4n - 1 or 4n + 1 where n is an integer. However, not every number expressed by 4n + 1 or 4n - 1 is prime.

Euler also discovered that every natural number can be expressed as the sum of no more than four squares of other whole numbers. Interesting designs may be made using beads, ceramics, cloth, etc.



DESIGN USING EULER'S DISCOVERY

ACTIVITIES

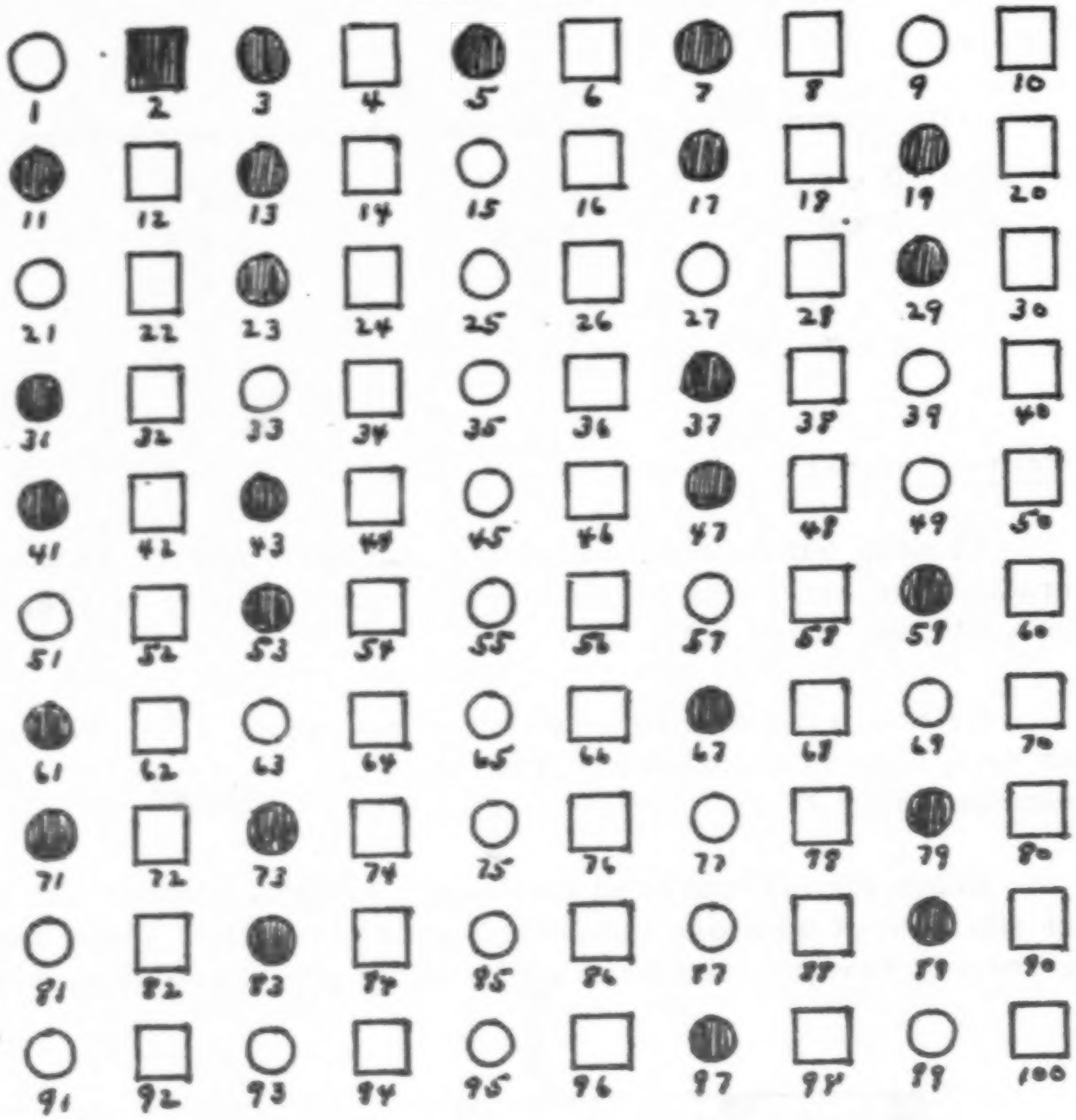
8.10 A Hundred Board

A Hundred Board can be made to show prime and composite numbers and even and odd numbers.

Circles may be used for odd numbers and squares for even numbers. Then prime numbers may be represented by colored circles or squares.

This board may be used as any hundred board in teaching number facts, but it has the advantage that prime and composite numbers stand out as well as showing the pattern of odd and even numbers.

A HUNDRED BOARD


1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100

8.11 Googols and Googolplexes

The nine year old nephew of Dr. Edward Kasner, a professor of mathematics, gave the name googol to the number 10^{100}. This would be 10 followed by a hundred zeros. Archmides calculated that the number of grains of fine sand needed to fill the universe is 10^{63}, which is less than the googol. The googol is greater than the greatest numbers used in astronomy.

4 OF 4

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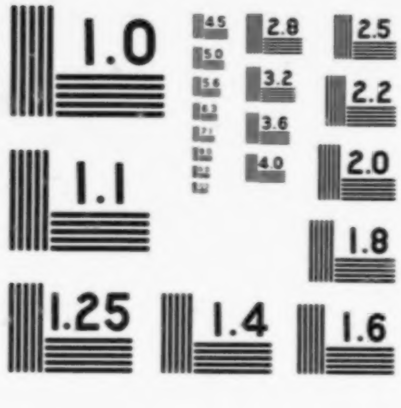

1.0
4.5
5.0
5.6
6.3
7.1
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963

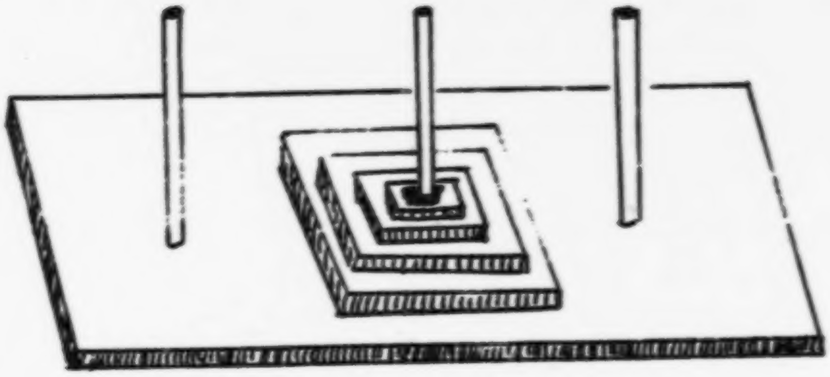
Vastly greater than a googol is a "googolplex" also named by Dr. Kasner's nephew. A googolplex is 10^{googol}. (See Kasner and Newman, Mathematics and the Imagination, New York: Simon and Schuster, 1940)

8.12 Tower of Hanoi

This is an old puzzle sometimes called the "End of the World Puzzle." The story is told that beneath the dome in the temple at Benares which marks the center of the world, there is a brass plate in which are fixed 3 diamond needles each a cubit (20 in.) high. On one of these needles, God placed 64 gold discs when he created the world. Each disc gets successively smaller and smaller up to the top one. The priest on duty in the tower of Brahma must transfer these discs from one needle to another, moving them according to the Brahman law which requires that the priest move only one disc at a time and that he never place a larger disc on a smaller. The story continues that when all the discs have been transferred to another needle, the tower, temple, and Brahmans will crumble to dust and the world will vanish with a loud thunderclap.

The least number of moves required is $2^{64} - 1$. If a move is made every second, 58 thousand billion years would be required to accomplish this task.

A similar puzzle may be constructed using only 5 discs as shown below. The number of moves needed is $2^5 - 1$ or 31.



TOWER OF HANOI

8.13 Odd Number Trick

Have a child divide an odd number of pennies, coins, buttons, or some chosen objects between the right and left hands. There will be an odd number in one hand and an even number in the other.

Then have the child multiply what he has in his right hand by 2 and what he has in his left hand by 3 and add the two products.

If the sum is even, the right hand has the odd number and if the sum is odd, the left hand has the odd amount.

14 Famous Mathematicians

1850 - 1935	Sonya Kovalevsky, a Russian mathematician, taught mathematics at the University of Stockholm. Her work on the theory of differential equations places her among the ranks of leading mathematicians. She was a pupil of Weierstrass and took her doctor's degree at Göttingen.
1858 - 1932	Guiseppi Peano, an Italian mathematician, was one of the creators of modern mathematical logic. He developed the natural number system from the undefined terms "natural number", "successor of", and "1" and from five axioms, the first of which is "1 is a natural number".
1862 - 1943	David Hilbert, a German mathematician, wrote a famous book on the foundations of geometry. He made an analysis of the axioms upon which Euclidean geometry is based and introduced more rigor into geometry than had formerly characterized it.
1879 - 1955	Albert Einstein, who spent his later years at the Institute for Advanced Study at Princeton, maintained that he invented the theory of relativity by challenging the axiom: Of two distinct instants, one must precede the other. It may be of interest to teachers to know that he was considered to be a very dull student during his early years of schooling. When he was 15 or 16 he was expelled from school because of his poor marks and his poor attitude. He did not really show promise until he was 22 years old.
1887 - 1920	Srinivasa Ramanujan, a mathematician from India, practically rediscovered for himself the work of three centuries in mathematics after he was denied entrance to the University of Madras because of his weakness in English. He finally came to the attention of Cambridge, where he was recognized as a self-taught genius.
1903 - 1957	John von Neuman, one of the mathematicians who opened up the theory of games which is used in solving problems in economics, science, and military strategy, had mastered college calculus at the age of 8. In 1930 he came to the United States from Budapest to become a professor of mathematical physics at Princeton University. In addition to helping build one of the first modern electronic computers, called MANIAC, he was influential in designing nuclear weapons and missiles during World War II.

EXERCISES

UNIT VIII

1. a. Draw a picture of a four hour clock.
   b. Use the clock to make addition and multiplication tables for a finite number system, modulo 4.

2. Answer the following questions if a modulo 4 system is being used.

   a. $2 + 1 = ?$
   b. $1 + 2 = ?$
   c. $0 + 3 = ?$
   d. $3 \times 3 = ?$
   e. $(2 \times 3) \times 2 = ?$
   f. $2 \times (3 \times 2) = ?$
   g. $(2 + 3) + 2 = ?$
   h. $2 + (3 + 2) = ?$
   i. $0 \times 3 = ?$
   j. $2 \times 2 = ?$
   k. $3 \times (2 + 3) = ?$
   l. $(3 \times 2) + (3 \times 3) = ?$
   * m. $3 - 2 = ?$
   * n. $2 \div 2 = ?$

*To subtract count in a counterclockwise direction or think of "what number needs to be added?" Then $2 - 3 \equiv ?$ (mod 4). Can you discover how to solve division problems?

3. Check the following for divisibility by the number in parenthesis at the right.

   a. 1468, 232, 3946 (4)
   b. 3281, 14684, 52691 (11)
   c. 81468, 2196, 7382 (9)

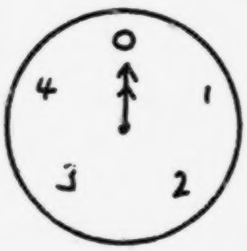
4. Find the prime factors of the following

   a. 462
   b. 96

5. Find the G. C. D. for the following sets of numbers using the intersection of sets of prime factors.

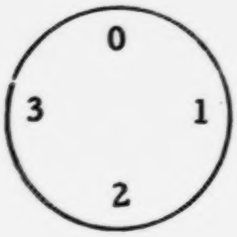
   a. 15, 33, 12
   b. 42, 56, 54

6. Suppose there were a clock having 5 numerals as shown at the left but with 2 hands--a long one and a short one. Both hands rotate at the same speed, but the long one rotates clockwise and the short one rotates counterclockwise. In order to tell time, add the two numbers to which the clock hands point. Assuming that this clock starts to run with both hands initially at 0 and that addition is commutative what will the time always be?



EXERCISES (answers) **UNIT VIII**

1. a.



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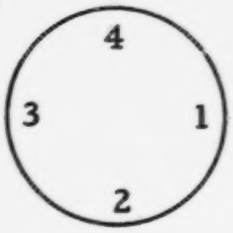


b.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

X	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. a. 3
b. 3
c. 3
d. 1
e. 0
f. 0
g. 3
h. 3
i. 0
j. 0
k. 3
l. 3
m. 1
n. No unique answer. Will this always be true if modulus is a composite number?

A division problem like 2 ÷ 2 can be thought of as 2 x ? = 2.

3. a. Yes, yes, no
b. No, no, no
c. Yes, yes, no

4. a. $2 \cdot 3 \cdot 7 \cdot 11$
b. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

5. a. $A = \{3, 5\}$ $B = \{3, 11\}$ $C = \{2, 2, 3\}$ $A \cap B \cap C = \{3\}$

b. $A = \{2, 3, 7\}$ $B = \{2, 2, 2, 7\}$ $C = \{3, 3, 3, 2\}$ $A \cap B \cap C = \{2\}$

6. 5 unless both stop at 0.

ASSIGNMENT

1. Determine whether the following numbers are prime or composite:

   a. 159
   b. 217
   c. 314
   d. 32157
   e. 163

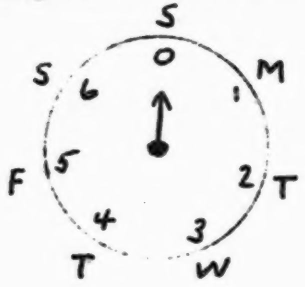
2. Check the following numbers for divisibility by the number represented by the numeral in the parentheses at the right:

   a. 4281; 38745; 14628 (9)
   b. 324; 146; 8214 (6)
   c. 126; 382; 322 (7)
   d. 2002; 42681; 332156 (11)

3. Develop a rule testing the divisibility of a number by 25; by 125.

4. Arrange the numerals 0, 1, 2, and 3 to represent the two numbers which are divisible by all of the numbers 2, 3, 4, 5, and 10.

5. Using the clock face at the left below answer the following :



   a. Prepare an addition table modulo 7
   b. Prepare a multiplication table modulo 7.
   c. 2 + 3 = ?
   d. 3 + 2 = ?
   e. 6 + 4 = ?
   f. 4 + 6 = ?
   g. 4 x 4 = ?
   h. (3 x 2) x 4 = ?
   i. 3 x (2 x 4) = ?
   j. 3 x (4 + 5) = ?
   k. (3 x 4) + (3 x 5) = ?
   l. (6 + 4) + 5 = ?

   *m. $x + 2 \equiv 3 \pmod{7}$
   *n. $x + 4 \equiv 2 \pmod{7}$
   o. 24 days after Monday is ———.
   p. 12 days before Thursday is ———.
   q. 16 days after Friday is ———.
   r. Today is Saturday, April 8. Helen's birthday is May 8. On what day of the week will she celebrate her birthday?

6. If it is 2:00 p.m. now, answer the following questions making use of congruences:

   a. What time will it be in 20 hours?
   b. What time will it be in 120 hours?
   c. What time will it be in 39 hours?
   d. What time will it be in 234 hours?

JUST FOR FUN

7. Develop a rule testing the divisibility of a number by 11 similar to the one for divisibility by 7.

8. To find the G. C. D. of a pair of natural numbers, it is convenient to use a method devised by Euclid and often referred to as the Euclidean algorithm for finding the G. C. D. For example, to find the G. C. D. of 48 and 264 using this algorithm first divide 264 by 48. Then divide 48 by the remainder. Continue to divide by remainders until an exact divisor is found. The exact divisor will be the G. C. D.

$$\begin{array}{rrl} & 5 & \\ 48 & \overline{)264} & \\ & \underline{240} & 2 \\ & 24 & \overline{)48} \quad \text{24 is G. C. D.} \\ & & \underline{48} \\ & & 0 \end{array}$$

Using this method find the GCD of the following pairs:

a. 28, 4263
b. 34, 2431
c. 92, 1532

9. Al, Bob, Chuck, Dick, and Ed were members of a basketball squad. The coach posted a paper with these instructions as to who would play and who would not play.

a. If Ed does not play, then Al and Bob will play.
b. Either Al or Bob will play, but not both.
c. If Ed plays, then Bob will play but Dick will not.
d. Dick will play if and only if Chuck does not play.

Find who played and who didn't.

10. 1/7 = .142857142857... and 1/13 = .076923076923...

a. 2 x 142857 =
b. 3 x 142857 =
c. 4 x 142857 =
d. 5 x 142857 =
e. 6 x 142857 =

f. 3 x .076923 =
g. 4 x .076923 =
h. 9 x .076923 =
i. 12 x. .076923 =

What do you observe about the answers when you multiply these cyclic numbers by the indicated numbers? Can you figure out which numbers to multiply by?

1. a. composite
   b. composite
   c. composite
   d. composite
   e. prime

2. a. No; yes; no
   b. Yes; no; yes
   c. Yes; no; yes
   d. Yes; no; yes

3. a. If number represented by last 2 digits of its numeral is divisible by 25
   b. If number represented by last 3 digits of its numeral is divisible by 125

4. 1320, 3120

5. a.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

b.

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

c. 5
d. 5
e. 3
f. 3
g. 2
h. 3
i. 3
j. 6
k. 6
l. 1
m. 1
n. 5
o. Thursday
p. Saturday
q. Sunday
r. Monday

6. a. 10 a. m.
   b. 2:00 p. m.
   c. 5:00 a. m.
   d. 8:00 a. m.

7. Subtract a number represented by the last digit of the numeral from the number represented by the remaining digits. If this difference is divisible by 11, the original number is. E. g. consider 341. If 34-1 is divisible by 11, 341 is.

ASSIGNMENT (answers)

8. a.

```
     152
28 ⟌4263
     28
     146
     140
      63
      56   4
      ⑦ | 28
```

7 is GCD

b.

```
       71
34 ⟌2431
    238
      51
      34     2
      ⑰ |  34
            34
```

17 is GCD

c.

```
        16
92 ⟌1532
      92
      612
      552    1
       60 | 92
            60    1
            32 | 60
                 32     1
                 28  | 32
                       28   7
                       ④ | 28
                            28
```

4 is GCD

9.

Al	Bob	Chuck	Dick	Ed
0	X	X	0	X

10. a. 285714 d. 714285 g. .307692
b. 428571 e. 857142 h. .076923
c. 571428 f. .230769 i. .923076

The answers contain only those numerals appearing in the numeral representing the original number, and the numerals are always in the same order but start with different digits.

Multiply by those numbers which will give a product whose last digit will be one of the digits in the original numeral representation.

BIBLIOGRAPHY

Adler--The Giant Golden Book of Mathematics, Exploring the World of Numbers and Space, Golden Press, 1960. $3.95

Adler--Magic House of Numbers, Doubleday Anchor Co., 1957 $.50 (paperback)
John Day, 1957. $3.00

Adler--The New Mathematics, John Day, 1958. $3.75 Mentor Paperback $.55

Banks, J. H.--Elements of Mathematics, Allyn & Bacon, 1961. $5.75

C-2 Bulletin--Thinking in the Language of Mathematics, Office of the Superintendent of Public Instruction, Springfield, Ill., 1959
(free to teachers)

Dantzig, T.--Number, the Language of Science, Macmillan 1954 $6.50
Doubleday, 1954 $.95

Heath, R. V.--Mathemagic, Dover Publications, 1953 $1.00 (paperback)

Hogben, L.--The Wonderful World of Mathematics, Doubleday, 1955. $2.95

Johnson and Glenn--Computing Devices $.60
The Pythagorean Theorem $.56
Number Patterns $.56
Invitation to Mathematics $.56
Sets, Sentences, and Operations $.64
Understanding Numeration Systems $.60

Webster Publishing Co., 1960

Larsen--Binary Numbers, Grade 8
Check and Double Check, Grade 7
Excursions in Arithmetic, Grade 6
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Magic Squares, Circles and Stars, Grade 5
Number Games, Grade 7
Number Systems, Grade 7
Oddities of Numbers, Grade 7
Other Ways, Grade 7
Queries and Queeries, Grade 8
This and That, Grade 8
Ways to Multiply, Grade 5

Row Peterson--Enrichment Program for Arithmetic, 1959
$1.20 per set of 8 for each grade (school price)

Merrill, Helen--Excursions in Mathematics, Dover Publications, 1957
$1.00 (paperback)

NCTM 24th Yearbook--The Growth of Mathematical Ideas, Grades K-12, National Council of Teachers of Mathematics, 1960 $5.00 ($4.00 to members)

NCTM 25th Yearbook--Instruction in Arithmetic, National Council of Teachers of Mathematics, 1960 $4.50 ($3.50 to members)

Osborn, DeVault, Boyd, and Houston--Extending Mathematics Understanding, Charles E. Merrill Books, Inc., 1961
$6.75

Smith and Ginsburg--Numbers and Numerals, National Council of Teachers of Mathematics, 1959 $.35

Swain, Robert L.--Understanding Arithmetic, Holt, Rinehart and Winston 1960. $4.75

Titchmarsh--Mathematics for the General Reader, Doubleday, 1959. $.95

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PRONUNCIATION OF NAMES AND WORDS

These pronunciations were taken from Webster's Biographical Dictionary and from Webster's Unabridged Dictionary.

The pronunciation key is as follows:

ā	āle
a̍	cha̍otic
â	care
ă	ădd
ä	ärm
ȧ	ȧsk
a	account
ē	ēve
e̍	e̍vent
ĕ	ĕnd
e	silent
ẽ	makẽr
ī	īce
ĭ	ĭll
i	charity
ō	ōld
o̍	o̍bey
ô	ôrb
ŏ	ŏdd
o	connect
o͝o	fo͝ot
o͞o	fo͞od
oi	oil
ou	out
ū	cūbe
u̍	u̍nite
û	ûrn
ŭ	ŭp
u	circus
ü	menü

Biographical Names

Abel -- ä′bel

Ahmes -- ä′mĕs

al - Khawarizmi -- ăl′ko͝o·wä′rĭz·mē′

Apollonius -- ăp′ŏ lō′nĭ us

Archimedes -- är′kĭ mē′dēz

Aristotle -- ăr′ĭs tŏt′l

Aryabhata -- är′ya bŭt′a

Bachet -- bä shā′

Bhaskara -- bäs′kȧ rȧ

Bolyai -- bo̍′lyoi

Brahmagupta -- brŭ′mȧ·go͝op′tȧ

Cantor -- kän′to̍r

Cauchy -- kō shē′

Cavalieri -- kä′vä·lyâ′re̍

Cayley -- kā′lĭ

Copernicus -- ko̍ pûr′nĭ kus

Dedekind -- dā′dĕ kĭnt

Demoivre -- dẽ moi′vẽr

Descartes -- dā kȧrt′

Diophantus -- dī′o̍·făn tus

Einstein -- īn′stīn′

Eratosthenes -- ĕr′ȧ tŏs′the̍·nēz

Euclid -- ū′klĭd

Euler -- oi′lẽr

Fermat -- fĕr mä′

Fibonnaci -- fē′bo᷵·nät′che᷵

Fourier -- fo͞o′ryā′

Galileo -- găl′ĭ lē′ō

Galois -- gä′lwä′

Gauss -- gous

Gerbert -- gĕr′bẽrt

Grassman -- gräss′män

Hermite -- ĕr′mēt′

Hilbert -- hĭl′bẽrt

Hipparchus -- hĭ pär′kus

Huygens -- hī′gĕnz

Hypatia -- hī pā′shĭ·ȧ

Kepler -- kĕp′lẽr

Khayyam -- kī yäm′

Kovalevsky -- kȧ′vȧ lyĕf′ske᷵

Lagrange -- lȧ gränzh′

La Hire -- lȧ ēr′

Laplace -- lȧ′plȧs′

Lebesgue -- lẽ·bâg′

Legendre -- lẽ·zhän′dr’

Leibniz or Leibnitz -- līp′nĭts

Levi-Civita -- lâ ve᷵·chē′ve᷵·tä

Lobachevski -- lŭ·bŭ·chāf′skû·ĭ

Leverrier -- lẽ vĕ′ryā

Liouville -- lyo͞o′vēl′

Maclaurin -- măk·lô′rĭn

Mahavira -- mȧ hä vē′rȧ

Mersenne -- mĕr′sĕn′

Mobius -- mû′be᷵ o͝os

Monge -- mônzh

Montucla -- môN tü′kla′

Napier -- nā′pĭ·ẽr

Newton -- nū′t'n

Paganini -- pä′gä nē′ne᷵

Pappus -- păp′us

Pasch -- päsh

Peano -- pa᷵·ä′no᷵

Poincare -- pwăN′kȧ′rā′

Poncelet -- pôns′lĕ′

Pythagoras -- pĭ·thăg′o᷵·rȧs

Ramanujan -- rä·mä′no͝o jŭn

Riemann -- rē′män

Srinivasa Ramanujan -- shrē′nĭ vä′sȧ

Steiner -- shtī′ner

Stevin -- sta᷵ vĭn′

Sylvester -- sĭl vĕs′tẽr

Thales -- thā′lēz

Vieta -- vyā′tä

Weierstrass -- vī′ẽr shträs

Zeno -- zē′nō

Words Used

abacus -- ăb′ȧ·kŭs

amicable -- ăm′ĭ·ka·b'l

associative -- ă sō′shĭ·ā′tĭv

Benares -- bĕ·nä′rĕz

binary -- bī′nȧ·rĭ

Brahma -- brä′mȧ

Brahman -- brä′man

commutative -- kŏ·mū′tȧ·tĭv

distributive -- dĭs trĭb′u̝ tĭv

duodecimal -- du̝′o̝·dĕs′ĭ·mal

googol -- go͞o′gŏl

Hanoi -- hä′noi′

isomorphic -- ī·so̝·môr′fĭk

Maya -- mä′yȧ

Mayan -- mä′yăn

modular -- mŏd′‿u̝ lẽr

modulo -- mŏd′‿u̝ lō

multiplicative -- mŭl′tĭ·plĭ·kā′tĭv

quipu -- kē′po͞o or kwĭp′o͞o

quinary -- kwī′nȧ rĭ

sexagesimal -- sĕk′sȧ jĕs′ĭ·măl

ternary -- tûr′nȧ·rĭ

topology -- to̝·pŏl′o̝·jĭ

vigesimal -- vī·jĕs′ĭ·măl